Math Probability
Math 477 Rutgers
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Twelvefold Way

Suppose we want to distribute n balls into m bins and count the number of possible ways. The answer depends on the restriction of the type of assignments and whether we assume the balls, bins to be labelled or identical.

Labelled Balls into Labelled Bins

- **No restrictions:** If we have no restrictions on the number of balls in a bin, the count is simple m^n. Each of the ball can be chosen independently to be assigned to any of the m bins.
- **No two balls in the same bin:** In this case first ball can be assigned in any of the m bins, once the first ball is assigned, second has to be assigned to a different bin, so has $m-1$ options, once this has been done, third has $m-2$ options and so on. So the count is $m(m-1)\cdots(m-n+1)$
- **Every bin has to contain at least one ball:**

  We need to ball the set of balls $\{1, 2, \cdots, n$ into m parts and give some part to the first bin, some part to second bin and so on. The number of ways to partition the set is called the Stirling number $S(n, m)$ and the number of way to assign the partition to the bins is $m!$. So the total count is $m!S(m, n)$

Unlabelled Balls into Labelled Bins

- **No restrictions:** The is problem we finding solutions to $x_1 + x_2 + \cdots x_m = n$. And encoding them using $m-1$ separators and n zeroes we see that the count is $\binom{n+m-1}{n}$.
- **No two balls in the same bin:** We assign one ball or none to each bin and a total of n ball to all the bins. So the count is $\binom{m}{n}$.
- **Every bin has to contain at least one ball:** We first assign one ball to each bin. We are left with $n-m$ balls. Now we distribute these to the bins. Hence $\binom{n-m+m-1}{n-m}$ ways.

Labelled Balls into unlabelled Bins

- **No restrictions:** If we fill r bins, each such filling involves partitioning the set of labelled balls $\{1, 2, \cdots, n\}$ into r parts. We can choose as many bin as we want. So we get $\sum_{r=1}^{m} S(n, r)$

- **No two balls in the same bin:** If we have at least n bins, ie., $m \geq n$ we have exactly one way. Otherwise there is no way.
- **Every bin has to contain at least one ball:** This means all the m bins have to be used and we need to partition the set into m parts. So we have $S(n, m)$ ways for doing this.

Unlabelled Balls into unlabelled Bins

- **No restrictions:** Every assignment corresponds to a way to partition n into at most m parts. We call this the partition number $p_m(n)$.
- **No two balls in the same bin:** Same as the previous case. If we have at least n bins, ie., $m \geq n$ we have exactly one way. Otherwise there is no way.
- **Every bin has to contain at least one ball:** In this case we need partition of n into exactly k parts. Hence the count is $p_m(n) - p_{m-1}(n)$, we remove partitions that just use at most $m-1$ parts from the total set of partitions.