Math Probability
Math 477 Rutgers
Surya Teja Gavva

Probability Puzzles

Balls into bins

We have m balls and n balls. Every ball is randomly placed into one of the bins. What is the distribution of maximum number of balls contained in a bin?

Banach's matchbox problem

Suppose a mathematician carries two matchboxes at all times: one in his left pocket and one in his right. Each time he needs a match, he is equally likely to take it from either pocket. Suppose he reaches into his pocket and discovers for the first time that the box picked is empty. If it is assumed that each of the matchboxes originally contained N matches, what is the probability that there are exactly k matches in the other box?

Bertrand's ballot problem

In an election a candidate A receives p votes and candidate B receives q votes with $p > q$. What is the probability that A will be strictly ahead of B throughout the count?

Birthday Paradox

What is probability that in a set of n randomly chosen people, some pair of them will have the same birthday? How many people do you need to get a probability of p.?

Birthday Holidays

Labor laws in Erewhon require factory owners to give every worker a holiday whenever one of them has a birthday and to hire without discrimination on grounds of birthdays. Except for these holidays they work a 365-day year. The owners want to maximize the expected total number of man-days worked per year in a factory. How many workers do factories have in Erewhon?

Buffon's needle problem

Suppose we have a floor made of parallel strips of wood, each the same width, and we drop a needle onto the floor. What is the probability that the needle will lie across a line between two strips?

Coupon collector's problem

Given n coupons, how many coupons do you expect you need to draw with replacement before having drawn each coupon at least once?

Gambler's ruin

Player M has 1, and Player N has 2. Each play gives one of the players 1 from the other. Player M is enough better than Player N that he wins 2/3 of the plays. They play until one is bankrupt What is the chance that Player M wins?

German tank problem

Suppose you know there are a fixed unknown number N of objects. If you randomly sample m objects without replacement from this collection, how do you estimate N?

Littlewood-Offord

Given a set of vectors $\{v_1, v_2, \cdots v_n\}$ what is the probability that a random subset has sum equal to k.

Mabinogion Sheep problem

At time $t = 0$ there is a herd of sheep each of which is black or white. At each time $t = 1, 2, \cdots$ a sheep is selected at random, and a sheep of the opposite color (if one exists) is changed to be the same color as the selected sheep. At any time one may remove as many sheep (of either color) as one wishes from the flock. The problem is to do this in such a way as to maximize the expected final number of black sheep.

Monty Hall problem

Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

Three prisoners problem

Three prisoners, A, B and C, are in separate cells and sentenced to death. The governor has selected one of them at random to be pardoned. The warden knows which one is pardoned, but is not allowed to tell. Prisoner A begs the warden to let him know the identity of one of the others who are going to be executed. "If B is to be pardoned, give me C's name. If C is to be pardoned, give me B's name. And if I'm to be pardoned, flip a coin to decide whether to name B or C."
The warden tells A that B is to be executed. Prisoner A is pleased because he believes that his probability of surviving has gone up from 1/3 to 1/2, as it is now between him and C. Prisoner A secretly tells C the news, who is also pleased, because he reasons that A still has a chance of 1/3 to be the pardoned one, but his chance has gone up to 2/3. Which prisoner is correct?

Cardano

How many throws of a fair die do we need in order to have an even chance of at least one six?

Galileo

Suppose three dice are thrown and the three numbers obtained added. The total scores of 9, 10, 11, and 12 can all be obtained in six different combinations. Why then is a total score of 10 or 11 more likely than a total score of 9 or 12?

Newton-Pepys problem

A asserts that he will throw at least one six with six dice. B asserts that he will throw at least two sixes by throwing 12 dice. C asserts that he will throw at least three sixes by throwing 18 dice. Which of the three stands the best chance of carrying out his promise?

Pill puzzle

We are removing pills randomly from a jar. If the pill removed is the whole pill, we break into half pills, one half is consumed and other is returned to the jar, if the pill removed is a half pill, it is consumed. If we start with m pills, what is the expected number of half-pills when the last whole pill is removed from the jar?

De Mere - Problem of Dice

When a die is thrown four times, the probability of obtaining at least one six is a little more than 1/2. However, when two dice are thrown 24 times, the probability of getting at least one double-six is a little less than 1/2. Why

De Mere - Problem of Points

Two players A and B play a fair game such that the player who wins a total of n rounds first wins a prize. Suppose the game unexpectedly stops when A has won a total of r rounds and B has won a total of s rounds. How should the prize be divided between A and B?

In a common carnival game a player tosses a penny from a distance of about 5 feet onto the surface of a table ruled in I-inch squares. If the penny (i inch in diameter) falls entirely inside a square, the player receives 5 cents but does not get his penny back; otherwise he loses his penny. If the penny lands on the table, what is his chance to win?

Marriage problem

Consider a woman who has n men willing to marry her. If all the men showed up at once, she could order them and choose the best. Unfortunately for her, the men arrive one at a time and in random order. After dating each man for a short period of time, she must decide, before moving on to the next, whether or not to marry him. If she rejects his marriage proposal, she cannot recall him at a later time and, should she decide to marry, she will have to forego meeting the remaining suitors. What strategy should she adopt if she wants to maximize the probability of marrying the best suitor?

Sultan's Dowry problem

A sultan has granted a commoner a chance to marry one of his n daughters. The commoner will be presented with the daughters one at a time and, when each daughter is presented, the commoner will be told the daughter's dowry (which is fixed in advance). Upon being presented with a daughter, the commoner must immediately decide whether to accept or reject her (he is not allowed to return to a previously rejected daughter). However, the sultan will allow the marriage to take place only if the commoner picks the daughter with the overall highest dowry. Then what is the commoner's best strategy, assuming he knows nothing about the distribution of dowries.

Robbins' problem

Let $X_1, X_2, \cdots, X_n$ be independent, identically distributed random variables, uniform on $[0, 1]$. We observe the X_k's sequentially and must stop on exactly one of them. No recall of preceding observations is permitted. What stopping rule minimizes the expected rank of the selected observation, and what is its corresponding value?

Sunrise problem

What is the probability that sun will rise tomorrow?

St. Petersburg paradox

A casino offers a game of chance for a single player in which a fair coin is tossed at each stage. The initial stake starts at 2 dollars and is doubled every time heads appears. The first time tails appears, the game ends and the player wins whatever is in the pot. Thus the player wins 2 dollars if tails appears on the first toss, 4 dollars if heads appears on the first toss and tails on the second, 8 dollars if heads appears on the first two tosses and tails on the third, and so on. Mathematically, the player wins 2k dollars, where k equals number of tosses (k must be a whole number and greater than zero). What would be a fair price to pay the casino for entering the game?

Urn problems

An urn contains red socks and black balls. When two balls are drawn at random, the probability that both are red is 1/2

- How small can the number of socks in the drawer be?
- How small if the number of black socks is even?

Tennis problem

To encourage Elmer's promising tennis career, his father offers him a prize if he wins (at least) two tennis sets in a row in a three-set series to be played with his father and the club champion alternately: father-champion-father or champion-father-champion, according to Elmer's choice. The champion is a better player than Elmer's father. Which series should Elmer choose?

Jury problem

A three-man jury has two members each of whom independently has probability p of making the correct decision and a third member who flips a coin for each decision (majority rules) A one-man jury has probability p of making the correct decision. Which jury has the better probability of making the correct decision?

Waiting time

On the average, how many times must a die be thrown until one gets a 6?

Perfect hand

What is the chance that you are dealt a perfect hand (13 of one suit) in a game of bridge?

Tournament problems

1. A tennis tournament has 8 players. The number a player draws from a hat decides his first-round rung in the tournament ladder. What is the probability that second best player will be the runner-up? (Assume in every game, the better player wins and the players are all ranked differently)
2. In a tennis tournment where all players are equally likely to win any game, what is the probability that two players A, B will play against each other?

Duel problem

A, B, and C are to fight a three-cornered pistol duel. All know that A's chance of hitting his target is 0.3, C's is 0.5, and B never misses. They are to fire at their choice of target in succession in the order A, B, C, cyclically (but a hit man loses further turns and is no longer shot at) until only one man is left unhit. What should A's strategy be?
3. There are 64 teams who play single elimination tourna- ment, hence 6 rounds, and you have to predict all the winners in all 63 games. Your score is then computed as follows: 32 points for correctly predicting the final winner, 16 points for each correct finalist, and so on, down to 1 point for every correctly predicted winner for the first round. (The maximum number of points you can get is thus 192.) Knowing nothing about any team, you flip fair coins to decide every one of your 63 bets. Compute the expected number of points.

Returning times

1) If A has a votes and B has b votes where $a > b$. If the votes are drawn and tallied in a random order, what is probability that at some point both candidates have equal number of votes. How many times does this happen?
2) Players A and B match pennies N times They keep a tally of their gains and losses. After the first toss, what is the chance that at no time during the game will they be even?

Random chord

If a chord is selected at random on a fixed circle what is the probability that its length exceeds the radius of the circle?

The Hurried Duelers

Duels in the town of Discretion are rarely fatal. There, each contestant comes at a random moment between 5 A.M. and 6 A M on the appointed day and leaves exactly 5 minutes later, honor served, unless his opponent arrives within the time interval and then they fight. What fraction of duds lead to violence?

Counterfeit coins

1. The king's minter boxes his coins 100 to a box. In each box he puts I false coin. The king suspects the minter and from each of 100 boxes draws a random coin and has it tested. What is the chance the minter's peculations go undetected?
2. The king's minter boxes his coins n to a box. Each box contains m false coins. The king suspects the minter and randomly draws 1 coin from each of n boxes and has these tested. What is the chance that the sample of n coins contains exactly r false ones?

Drunkard Walk

From where he stands, one step toward the cliff would send the drunken man over the edge. He takes random steps, either toward or away from the cliff. At any step his probability of taking a step away is 2/3, of a step toward the cliff 1/3. What is his chance of escaping the cliff?

Principle of symmetry

When n points are dropped at random on an interval, the lengths of the $n+1$ line segments have identical distributions.
In a laboratory, each of a handful of thin 9-inch glass rods had one tip marked with a blue dot and the other with a red. When the laboratory assistant tripped and dropped them onto the concrete floor, many broke into three pieces. For these, what was the average length of the fragment with the blue dot?

Breaking a stick

a) If a stick is broken in two at random, what is the average length of the smaller piece?
(b) (For calculus students.) What is the average ratio of the smaller length to the larger?
c) A bar is broken at random in two places. Find the average size of the smallest, of the middle-sized, and of the largest pieces

Optimal play length for binomial game

A game consists of a sequence of plays; on each play either you or your opponent scores a point, you with probability p (less than 1/2), he with probability $1 - p$. The number of plays is to be even: 2 or 4 or 6 and so on. To win the game you must get more than half the points. You know p, say 0.45, and you get a prIze if you win. You get to choose in advance the number of plays. How many do you choose?

Matchings

A typist types letters and envelopes to n different persons. The letters are randomly put into the envelopes. On the average, how many letters are put into their own envelopes? what is the probability of exactly r matches?

Random Polynomials

What is the probability that a random quadratic equation $x^2 + ax + b$ has real roots?

Random walk

Starting from an origin 0, a particle has a $50 - 50$ chance of moving 1 step north or 1 step south, and also a $50 - 50$ chance of moving 1 step cast or 1 step west. After the step is taken, the move is repeated from the new position and so on indefinitely. What is the chance that the particle returns to the origin?

Lost Boarding Pass

One hundred people line up to board an airplane. Each has a boarding pass with assigned seat. However, the first person to board has lost his boarding pass and takes a random seat. After that, each person takes the assigned seat if it is unoccupied, and one of unoccupied seats at random otherwise. What is the probability that the last person to board gets to sit in his assigned seat?

Intersecting Intervals

Choose two random numbers from [0, 1] and let them be the endpoints of a random interval. Repeat this n times. What is the probability that there is an interval which intersects all others.

Pairing Socks

You are in possession of n pairs of socks (hence a total of $2n$ socks) ranging in shades of grey, labeled from 1 (white) to n (black). Take the socks blindly from a drawer and pair them at random. What is the probability that they are paired so that the colors of any pair differ by at most 1?

Arc containing the chosen point

Choose, at random, three points on a circle. Interpret them as cuts that divide the circle into three arcs. Compute the expected length of the arc that contains a fixed point P.
Here is a solution: Let L_1, L_2, L_3 be the length of the arcs. $L_1 + L_2 + L_3 = 2\pi$. By symmetry $E(L_1) = 2\pi/3$. Is this argument correct?

Enlosing Origin

Four points are chosen on the unit sphere. What is the probability that the origin lies inside the tetrahedron determined by the four points?

Simulating probabilities

Let $\alpha \in [0,1]$ be an arbitrary number, rational or irrational. The only randomizing device is an unfair coin, with probability $p \in (0,1)$ of heads. Design a game between Alice and Bob so that Alice's winning probability is exactly α. The game of course has to end in a finite number of tosses with probability 1.

Hats Puzzle

Three players enter a room and a red or blue hat is placed on each person's head. The color of each hat is determined by [an independent] coin toss. No communication of any sort is allowed, except for an initial strategy session before the game begins. Once they have had a chance to look at the other hats [but not their own], the players must simultaneously guess the color of their own hats or pass. The puzzle is to find a group strategy that maximizes the probability that at least one person guesses correctly and no-one guesses incorrectly

Using Randomness

Somebody chooses two nonnegative integers X and Y and secretly writes them on two sheets of paper. The distribution of (X, Y) is unknown to you, but you do know that X and Y are different with probability 1. You choose one of the sheets at random, and observe the number on it. Call this random number W and the other number, still unknown to you, Z. Your task is to guess whether W is bigger than Z or not. You have access to a random number generator, i.e., you can generate independent uniform (on $[0,1]$) random variables at will, so your strategy could be random. Exhibit a strategy for which the probability of being correct is $1/2 + \epsilon$, for some $\epsilon > 0$. This ϵ may depend on the distribution of (X, Y), but your strategy of course can not.

Closest Integer

Two real numbers X and Y are chosen at random in the interval $(0, 1)$. Compute the probability that the closest integer to X/Y is even.

World Series

You are a broker; your job is to accommodate your client's wishes without placing any of your personal capital at risk. Your client wishes to place an even $1,000$ bet on the outcome of the World Series, which is a baseball contest decided in favor of whichever of two teams first wins 4 games. That is, the client deposits his $1,000$ with you in advance of the series. At the end of the series he must receive from you either $2,000$ if his team wins, or nothing if his team loses. No market exists for bets on the entire world series. However, you can place even bets, in any amount, on each game individually. What is your strategy for placing bets on the individual games in order to achieve the cumulative result demanded by your client?

Elevator problem

Mr. Smith works on the 13th floor of a 15 floor building. The only elevator moves continuously through floors $1, 2, \cdots, 15, 14, \cdots 2, 1, 2, \cdots$, except that it stops on a floor on which the button has been pressed. Assume that time spent loading and unloading passengers is very small compared to the travelling time. Mr. Smith complains that at 5 pm, when he wants to go home, the elevator almost always goes up when it stops on his floor. What is the explanation?

Strings and loops

Start with n strings, which of course have $2n$ ends. Then randomly pair the ends and tie together each pair. (Therefore you join each of the n randomly chosen pairs.) Let L be the number of resulting loops. Compute $E(L)$.

Random Walk-Circle

You have $n > 1$ numbers $0, 1, \cdots, n-1$ arranged on a circle. A random walker starts at 0 and at each step moves at random to one of its two nearest neighbors. For each i, compute the probability p_i that, when the walker is at i for the first time, all other points have been previously visited, i.e., that i is the last new point.

De Moivre's Problem

A fair die is thrown n independent times. Find the probability of obtaining a sum equal to t, where t is a natural number.

Parameter Estimation

Assume that the probability p of an event A is equally likely to be any number between 0 and 1. Show that, conditional on A having previously occurred in a out of n independent and identical trials, the probability that p lies betweenp_1 and p_2 is?

Robotic Monkey

A robot monkey is seated at a typewriter and randomly hits the keys in an infinite sequence of independent trials. The monkey will eventually type out the complete works of Shakespeare with certainty!

Benford's Law

In many naturally occurring tables of numerical data, it is observed that the leading (i.e., leftmost) digit is not uniformly distributed among $\{1, 2, \cdots, 9\}$, but the smaller digits appear more frequently that the larger ones. In particular, the digit one leads with a frequency close to 30% whereas the digit nine leads with a frequency close to only 4%. Explain.

Arc Sine Law

Find the probability that
1. the last time there was an equal number of heads and tails within $2n$ tosses occurred at the $2k$-th toss.
2. the first time there was an equal number of heads and tails within $2n$ tosses occurred at the $2k$-th toss.

Parronda's paradox

Consider the following games G_1, G_2, and G_3, in each of which \$1 is won if a head is obtained, otherwise \$1 is lost (assuming the player starts with \$0).
G_1 : A biased coin that has probability of heads .495 is tossed.
G_2 : If the net gain is a multiple of three, coin A is tossed. The latter has probability of heads .095. If the net gain is not a multiple of three, coin B is tossed. The latter has probability of heads .745.
G_3 : G_1 and G_2 are played in any random order.
Prove that although G_1 and G_2 each result in a net expected loss, G_3 results in a net expected gain.