BERNSTEIN POLYNOMIALS

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The Bernstein polynomial $B_n(f)$ of a function f defined on $[0,1]$ is defined as

$$B_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f(\frac{k}{n})$$

Approximation theorem

Let f be a function defined on $[0,1]$. For each point x of continuity of f, $B_n(f)(x) \to f(x)$ as $n \to \infty$. If f is continuous on $[0,1]$, then the Bernstein polynomial $B_n(f)$ converges to f uniformly i.e., $\max_{x\in[0,1]} |f(x) - B_n(f)| \to 0$. Morever for x a point of differentiability of f, $B_n'(f)(x) \to f'(x)$ If f is continuously differentiable on $[0,1]$, then $B_n'(f)(x) \to f'(x)$ uniformly.

We have the following formulae

$$B_n(1) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = 1$$

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} (\frac{k}{n}) = x$$

$$B_n(x^2) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} (\frac{k}{n})^2 = \frac{n-1}{n} x^2 + \frac{x}{n}$$

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} (x - \frac{k}{n})^2 = \frac{x(1-x)}{n}$$

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} (x-\frac{k}{n})^4 = \frac{x(1-x)(1+x(1-x)(3n-6))}{n^3}$$

Probabilistic idea:

$\binom{n}{k}x^k(1-x)^{n-k}$ is the probability of getting k heads in n throws if the probability of getting head is x.– See Bernoulli distribution.

Hence the above expression can be interpreted as the expectation of the random variable $f(K/n)$, where K is Bernoulli variable with probability parameter x

The expected value of K is nx and by law of large numbers, the probability is mostly concentrated around $k \approx nx$. So $f(k/n)$ is almost likely $f(x)$. Hence the expected value which is $B_n(f)$ behaves like $f(x)$ for large n.

Proof of the theorem:

$$B_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f(\frac{k}{n})$$

$$f(x) = f(x) \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}$$

$$B_n(f)(x) - f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f(\frac{k}{n}) - \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f(x)$$

$$|B_n(f)(x) - f(x)| = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} |f(\frac{k}{n}) - f(x)|$$

We consider terms with $|\frac{k}{n} - x| < \delta$ and those with $|\frac{k}{n} - x| \geq \delta$ seperately. As discussed in the proof, the former contributes the most to the sum. δ is chosen as follows
Given $\varepsilon > 0$, for x a point of continuity we have
$|f(x) - f(y)| < \varepsilon$ if $|x - y| < \delta_\varepsilon$

$$\sum_{|\frac{k}{n}-x|<\delta} \binom{n}{k} x^k (1-x)^{n-k} |f(\frac{k}{n}) - f(x)|$$

$$< \sum_{|\frac{k}{n}-x|<\delta} \binom{n}{k} x^k (1-x)^{n-k} \varepsilon$$

$$< \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \varepsilon = \varepsilon$$

$$\sum_{|\frac{k}{n}-x|\geq\delta} \binom{n}{k} x^k(1-x)^{n-k}|f(\frac{k}{n})-f(x)|$$

$$< \sum_{|\frac{k}{n}-x|\geq\delta} \binom{n}{k} x^k(1-x)^{n-k}(2M)$$

$$< 2M \sum_{|\frac{k}{n}-x|\geq\delta} \frac{(x-\frac{k}{n})^2}{\delta^2} \binom{n}{k} x^k(1-x)^{n-k}$$

$$< 2M \sum_{k=0}^{n} \frac{(x-\frac{k}{n})^2}{\delta^2} \binom{n}{k} x^k(1-x)^{n-k}$$

$$= \frac{2Mx(1-x)}{n\delta^2} < \frac{2M}{n\delta^2}$$

Choosing n large enough so that $\frac{2M}{n\delta^2} < \varepsilon$ we have

$$|f(x) - B_n(f)| < 2\varepsilon$$

$\square$