November 22, 1995

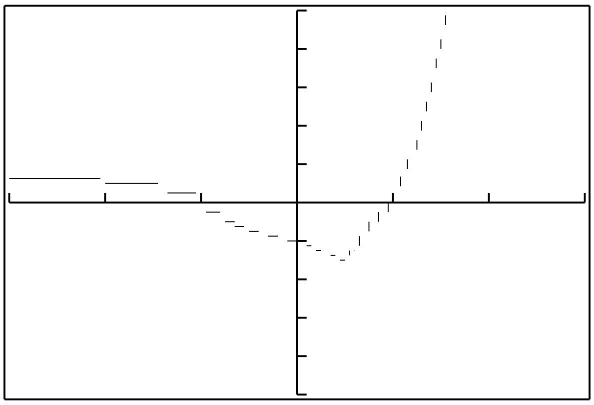
Answers to Exam #2 for Math 151, sections 31 & 32

Here are the questions together with some answers to the second exam. There are certainly other methods of getting the answers to some of the questions. I hope that these will be useful to you. Please note: I have added occasional **Comments** not really needed for the solutions which might be interesting or helpful.

(25) 1. Suppose $F(x) = (x^2 - 1)e^x$.

a) Use your calculator to sketch a graph of this function in the window with $-3 \le x \le 3$ and $-5 \le y \le 5$:

Answer



Comment This is my honest attempt to reproduce what the TI-82 actually displays as the graph of this function in this window. Drawing this figure was quite difficult, but I tried very hard to make it authentic in order to emphasize that the calculator, after all, only *approximates* the geometry of the curve: it computes various values of the function designated, and then rounds off the values and turns on and off various "pixels" in the screen to graph the function. Please read more about this in my comment after the graph below.

b) Verify the details of the graph displayed using the analytic tools of calculus. Be sure to answer all of the questions below:

Comment As I mentioned in class during the exam, darn it, I should have been more precise about the domain: I *did* want you to analyze the function in all of the real numbers, not just in the windowed area above. I regret the lack of clarity in the question.

Where is $F(x) = 0$? **Answer** The exponential function is never 0, so here $x = \pm 1$.

What is $\lim_{x \to \infty} F(x)$? Why? **Answer** $+\infty$, because both factors in F get large as $x \to \infty$.

What is $\lim_{x \to -\infty} F(x)$? Why? **Answer** $\lim_{x \to -\infty} (x^2 - 1)e^x = \lim_{x \to -\infty} \frac{x^2 - 1}{e^{-x}} \overset{\text{l'Hosp}}{=} \lim_{x \to -\infty} \frac{2x}{e^{-x}} \overset{\text{l'Hosp}}{=} \lim_{x \to -\infty} \frac{2}{e^{-x}} = 0.$

Comment I should have asked if the graph had either vertical or horizontal asymptotes, and this computation would have signaled to you that the $x-$axis ($y = 0$) was a horizontal asymptote.

What is $F'(x)$? **Answer** $F'(x) = 2xe^x + (x^2 - 1)e^x = (x^2 + 2x - 1)e^x$.

Where is $F'(x) = 0$? **Answer** The exponential function is never 0, so here $x = \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$.

Does F have any local extrema? If there are local extrema, where are they and what are they? Explain your conclusions in complete English sentences; express your answers exactly, algebraically, not with computational approximations.

Answer Yes. F has a local minimum at $-1 - \sqrt{2}$, since there F' changes sign from negative to positive (as x increases). Also, F has a local maximum at $-1 + \sqrt{2}$, since there F' changes sign from positive to negative (as x increases).

What are the largest intervals on which F is increasing? Express your answers exactly, algebraically, not with computational approximations. **Answer** $(-\infty, -1 - \sqrt{2}]$ and $[-1 + \sqrt{2}, \infty)$.

What are the largest intervals on which F is decreasing? Express your answers exactly, algebraically, not with computational approximations. **Answer** $[-1-\sqrt{2},-1+\sqrt{2}]$.

What is $F''(x)$? **Answer** $F''(x) = (2x+2)e^x + (x^2+2x-1)e^x = (x^2+4x+1)e^x$.

Where is $F''(x) = 0$? **Answer** The exponential function is never 0, so here $x = \dfrac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \sqrt{3}$.

Does the graph of F have any points of inflection? If there are any, where are they? Explain your conclusions; express your answers exactly, algebraically, not with computational approximations.

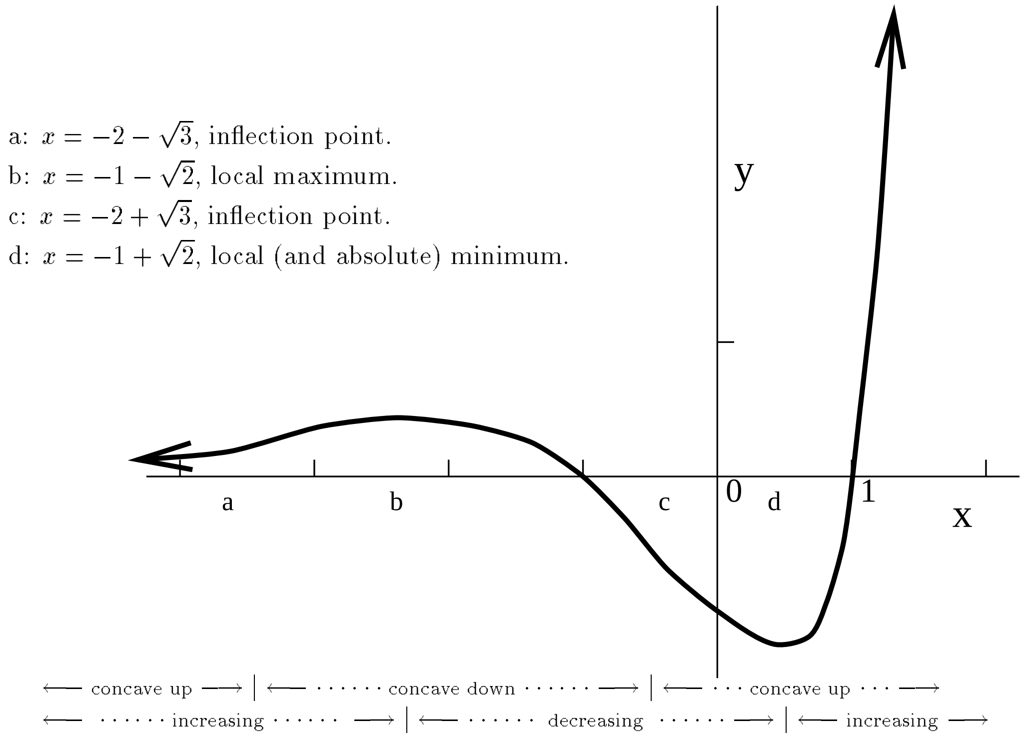
Answer Yes. F has points of inflection at $-2-\sqrt{3}$ and at $-2+\sqrt{3}$ because at both numbers F'' changes sign (as x increases).

What are the largest intervals in which the graph of F is concave up? Express your answers exactly, algebraically, not with computational approximations. **Answer** $(-\infty,-2-\sqrt{3}]$ and $[-2+\sqrt{3},\infty)$.

What are the largest intervals in which the graph of F is concave down? Express your answers exactly, algebraically, not with computational approximations. **Answer** $[-2-\sqrt{3},-2+\sqrt{3}]$.

Make *another* sketch of the graph of $y = F(x)$. Choose your axes carefully and label the graph in order to exhibit *all* of the phenomena found.

Answer



Comment Using a program I tried to draw my impression of what the graph of $y = F(x)$ looks like. I drew it stylistically rather than realistically. There are programs, very analogous to the TI-82's display facilities, which digitize and display. But almost all of the pictures I have drawn for this course have been done "freehand" — I have tried to convey the important aspects of the graphs by (frequently!) overemphasizing these aspects. This approach is rather familiar to anyone who has ever watched and then tried to identify birds. The most helpful field guides are *not* collections of authentic photographs of the creatures, because eyes tend naturally to grab and emphasize details that film does not. The most helpful illustrations are drawings where the important details are overcolored or signaled in various ways. Similarly, the pictures remembered of functions tend not to be the exact truth, but rather drawings emphasizing the important details (such as local extrema and points of inflection). Calculators and computers are useful, but are no true and total substitute for the eye, the brain, and the hand (at least, yet!).

(15) 2. Suppose that $C(4) = 5$ and $C'(x) = \sqrt{3x^2+1}$. If $B(x) = x^2$, and if $A(x) = C(B(x))$, what is $A(2)$? Write a formula for $A'(x)$ only in terms of x, and use this formula to find $A'(2)$.

Answer $A(2) = C(B(2)) = C(2^2) = C(4) = 5$. By the chain rule,

$$A'(x) = C'(B(x)) \cdot B'(x) = \left(\sqrt{3\,(B(x))^2 + 1}\ \right)(2x) = \left(\sqrt{3\,(x^2)^2 + 1}\ \right)(2x) = \left(\sqrt{3x^4+1}\right)(2x)$$

and therefore $A'(2) = \left(\sqrt{3 \cdot 2^4 + 1}\right)(2 \cdot 2) = \sqrt{49} \cdot 4 = 28$. (Whew! At least I got it to be an integer!)

a) Use the number you have found to write an equation for the line tangent to the graph of $y = A(x)$ when $x = 2$.

Answer The slope is 28 and the line goes through the point $(2, 5)$, so one equation for the line is

$$(y - 5) = 28(x - 2)\ .$$

b) Find the linear approximation to the function $A(x)$ at $x = 2$. Use this approximation to approximate the number $A(2.1)$.

Answer The linear approximation is the function whose graph is the tangent line, so the linear approximation is $5 + 28(x - 2)$ and when $x = 2.1$ this gives $5 + 28(2.1 - 2) = 7.8$ as the value desired.

Comment As we will see next semester, a formula for $A(x)$ *can* be gotten. Here is one:

$$A(x) = \frac{1}{2}x^2\sqrt{3x^4+1} + \frac{1}{6}\ln\left(\sqrt{3}x^2 + \sqrt{3x^4+1}\right) - 9.76034600$$

and then you can use this formula to actually compute $A(2.1)$ (you'll get 8.0142, approximately) instead of the value 7.8 given above. Several questions arise:

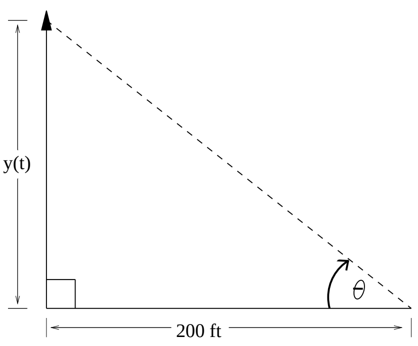
i) Who cares? **ii)** Is the formula correct? **iii)** How the heck do you get such a formula?

The easiest of these questions to answer is the last — it turns out that there is a straightforward and rather dull procedure to obtain such formulas. We'll discuss it in a few more months. You can answer the middle question yourself, if you dare: just differentiate and try to match the answer with the $A'(x)$ given above. The first question is too profound for me.

(15) 3. A rocket launched vertically upward from the ground is being tracked by radar located on the ground 200 feet from the point of lift-off.

a) If θ is the radar's elevation angle, i.e., the angle between the ground and the rocket as seen from the position of the radar sight, express the height $y(t)$ of the rocket as a function of θ.

Answer



Clearly $\tan\theta = \dfrac{y(t)}{200}$ so $y(t) = 200\tan\theta$.

b) At the moment the radar's elevation angle is $\dfrac{\pi}{3}$, the angle is increasing at the rate of .7 radians/second. How fast is the rocket rising at that moment?

Answer Just $\dfrac{d}{dt}$ the last equation above, being careful to use the chain rule:

$$\frac{dy}{dt} = 200\,(\sec\theta)^2\,\frac{d\theta}{dt}$$

When $\theta = \frac{\pi}{3}$, we know $\sec\theta = 2$ and $\frac{d\theta}{dt} = .7$ so $\frac{dy}{dt} = 200 \cdot 2^2 \cdot .7 = 560$ feet/sec.

Comment I didn't ask you for the units in the answer above. I probably should have. And of course you didn't *have* to draw a picture, but I find a picture very useful for keeping track of everything.

(15) 4. Suppose Q is a function satisfying the following differential equation:

$$Q'(x) = \frac{1}{1+Q(x)^3}$$

and we know that $Q(1) = 2$. What is $Q'(1)$?

Answer Since $Q(1) = 2$, $Q'(1) = \frac{1}{1+2^3} = \frac{1}{9}$.

a) If R is a function *inverse* to Q, find an equation for $R'(w)$ in terms of w only. (Hint: what is $Q(R(w))$? Differentiate the resulting equation.)

Answer We $\frac{d}{dw}$ the equation $Q(R(w)) = w$ and get: $Q'(R(w)) \cdot R'(w) = 1$ (not forgetting the chain rule). But $Q'(R(w)) = \frac{1}{1+Q(R(w))^3} = \frac{1}{1+w^3}$, where the first equation holds by what we were told at the beginning of the problem and the second equation holds because Q and R are inverse functions. Thus $R'(w) = 1 + w^3$.

b) What are $R(2)$ and $R'(2)$?

Answer $R(2) = 1$ because $Q(1) = 2$, and $R'(2) = 1 + 2^3 = 9$ using the formula developed in a).

Extra credit, **5 points** c) In this case due to the simplicity of the formula you have found in a) for $R'(w)$ it is possible to write an explicit formula for $R(w)$ which must also fit the data you have gotten in b). Find such a formula.

Answer We know that $R'(w) = 1 + w^3$. One of the applications of the Mean Value Theorem was to tell us *all* functions whose derivative is a given function. Here's how: just "guess" one such function. Then all of the functions are the specific guess + a constant. We can make a rather well-educated guess in this case: $R(w) = w + \frac{1}{4}w^4 + C$ where C is a constant which can be determined because we know that $R(2) = 1$. Thus when $w = 2$ our guess becomes: $2 + \frac{1}{4}2^4 + C = 1$ so that C must be -5. And $R(w) = w + \frac{1}{4}w^4 - 5$.

(15) 5. Suppose we know the following information about the function S:

$S(1) = 2$

If $1 \le x \le 3$ then $x + 1 \le S'(x)$.

If $1 \le x \le 3$ then $S'(x) \le 1 + x^2$.

a) What is the *smallest* value $S'(x)$ could have in the interval $1 \le x \le 3$?

Answer The smallest that $x+1$ could be on the interval $[1,3]$ is 2, and since $x + 1 \le S'(x)$ on that interval, $S'(x) \ge 2$ on $[1.3]$.

b) What is the *largest* value $S'(x)$ could have in the interval $1 \le x \le 3$?

Answer The largest that $1+x^2$ could be on the interval $[1,3]$ is 10, and since $S'(x) \le 1+x^2$ on that interval, $S'(x) \le 10$ on $[1.3]$.

c) What is the range of possible values for $S(3)$? Justify your result.

Answer We will *use* the answers to a) and b) in this section. Perhaps it was a mistake on my part not to indicate precisely that this could or should be done! What do we know?

$$S(1) = 2 \qquad\qquad 2 \le S'(x) \le 10 \text{ on } [1,3]$$

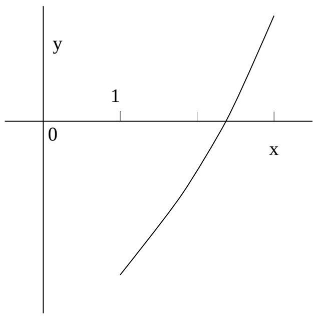
This is a setup for a simple application of the Mean Value Theorem. Thus $\dfrac{S(3)-S(1)}{3-1} = S'(x)$ for some x in $[1,3]$. "Unrolling" this equation we get

$$S(3) = S(1) + S'(x)(3-1)$$

We can easily plug in numbers now. The largest $S(3)$ can be is $2 + 10 \cdot 2 = 22$ using the answer to b), and the smallest $S(3)$ can be is $2 + 2 \cdot 2 = 6$ using the answer to a). The range of possible values for $S(3)$ consists of the interval $[6, 22]$.

(15) 6. a) Sketch the graph of $f(x) = x\ln(x) - 2$ on the interval $[1,3]$ to verify that f has a root in this interval.

Answer



Maybe I should just have asked you to evaluate $f(1) = -2$ and $f(3) \approx 1.2958$ and conclude by using the Intermediate Value Theorem that f has a root in $[1,3]$.

b) If Newton's method is applied to find a root of f, we obtain a sequence of numbers defined by a formula of the form $x_{n+1} = g(x_n)$. What is the function $g(x)$ when $f(x)$ is the function in a)?

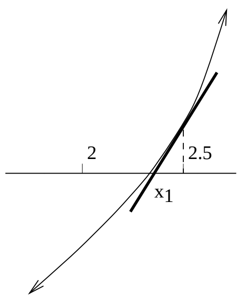
Answer Of course g is given by $x - \dfrac{f(x)}{f'(x)}$. Specifically $g(x) = x - \left(\dfrac{x\ln(x) - 2}{\ln(x) + 1}\right)$.

c) If the starting guess for Newton's method is taken to be 2.5, find the next approximation x_1.

Answer I think $g(2.5) \approx 2.34828$. This is x_1. An answer to five digit accuracy is 2.34575, a number gotten by Newton's method at the next iteration. Newton's method is *fast.*

d) Display graphically how Newton's method obtains the value x_1. That is, on a single graph for $1 \le x \le 3$, sketch $f(x)$ and the curve whose intersection with the x-axis is the number x_1.

Answer



EXAM 2 for MATH 151, sections 31 and 32

November 21, 1995

NAME (*please print*): ______________________________

SIGNATURE: ______________________________

Please circle your section:

31 **32**

Do all problems, in any order.

Show all your work. Full credit may not be given for an answer alone.

You may use one sheet of notes and any standard calculator without a QWERTY keypad on this exam. You may use no other materials.

Problem Number	Possible Points	Points Earned:
1	25	
2	15	
3	15	
4	15	
5	15	
6	15	
Total Points Earned:		