Combinatorial Auctions Do Need Modest Interaction

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We study the necessity of interaction for obtaining efficient allocations in combinatorial auctions with *subadditive bidders*. This problem was originally introduced by Dobzinski, Nisan, and Oren (STOC'14) as the following simple market scenario: m items are to be allocated among n bidders in a distributed setting where bidders valuations are private and hence *communication* is needed to obtain an efficient allocation. The communication happens in rounds: in each round, each bidder, *simultaneously* with others, broadcasts a message to all parties involved. At the end, the central planner computes an allocation solely based on the communicated messages. Dobzinski et al. showed that (at least some) interaction is necessary for obtaining any efficient allocation: no non-interactive (1-round) protocol with polynomial communication (in the number of items and bidders) can achieve approximation ratio better than $\Omega(m^{1/4})$, while for any $r \geq 1$, there exists r-round protocols that achieve $\widetilde{O}(r \cdot m^{1/r+1})$ approximation with polynomial communication; in particular, $O(\log m)$ rounds of interaction *suffice* to obtain an (almost) efficient allocation, i.e., a polylog(m)-approximation.

A natural question at this point is to identify the "right" level of interaction (i.e., number of rounds) *necessary* to obtain an efficient allocation. In this paper, we resolve this question by providing an almost tight *round-approximation* tradeoff for this problem: we show that for any $r \geq 1$, any r-round protocol that uses poly(m, n) bits of communication can only approximate the social welfare up to a factor of $\Omega(\frac{1}{r} \cdot m^{1/2r+1})$. This in particular implies that $\Omega(\frac{\log m}{\log\log m})$ rounds of interaction are *necessary* for obtaining any efficient allocation (i.e., a constant or even a polylog(m)-approximation) in these markets. Our work builds on the recent multi-party round-elimination technique of Alon, Nisan, Raz, and Weinstein (FOCS'15) – used to prove similar-in-spirit lower bounds for round-approximation tradeoff in unit-demand (matching) markets – and settles an open question posed initially by Dobzinski et al., and subsequently by Alon et al.

CCS Concepts: • **Theory of computation → Algorithmic mechanism design**; *Communication complexity*;

Additional Key Words and Phrases: combinatorial auctions, multi-party communication complexity, round-approximation tradeoff

1 INTRODUCTION

In a combinatorial auction, m items in M are to be allocated between n bidders (or players[1]) in N with valuation functions $v_i : 2^M \rightarrow \mathbb{R}_+$. The goal is to find a collection of disjoint bundles $A_1, \ldots, A_n$ of items in M (an *allocation*), that maximizes *social welfare* defined as the sum of bidder's valuations for the allocated bundles, i.e., $\sum_{i \in N} v_i(A_i)$. We study the tradeoff between the amount of interaction between the bidders and the efficiency of the allocation in combinatorial auctions.

In our model, each bidder $i \in N$ only knows the valuation function v_i and hence the bidders need to communicate to obtain an efficient allocation. Communication happens in rounds. In each

[1]Throughout the paper, we use the terms "bidder" and "player" interchangeably.

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round, each bidder i, *simultaneously* with others, broadcasts a message to all parties involved, based on the valuation function v_i and messages in previous rounds. In the last round, the central planner outputs the allocation solely based on the communicated messages. Notice that a "trivial solution" in this setting is for all players to communicate their entire input to the central planner who can then compute an efficient allocation; however, such a protocol is clearly infeasible in most settings as it has an enormous communication cost. As such, we are interested in protocols with significantly less communication cost, typically *exponentially smaller* than the input size.

This model was first introduced by Dobzinski, Nisan, and Oren [10] to address the following fundamental question in economics: *"To what extent is interaction between individuals required in order to efficiently allocate resources between themselves?"*. They considered this problem for two different classes of valuation functions: *unit-demand* valuations and *subadditive* valuations (see Section 2.1). For both settings, they showed that (at least some) interaction is necessary to obtain an efficient allocation: non-interactive (aka 1-round or simultaneous) protocols have enormous communication cost compared to interactive ones, while even allowing a modest amount of interaction allows for finding an (approximately) efficient allocation. We now elaborate more on these results.

For the case of matching markets with n unit-demand bidders and n items (and hence input-size of n bits per each player), Dobzinski et al. [10] proved a lower bound of $\Omega(\sqrt{n})$ on the approximation ratio of any simultaneous protocol that communicates $n^{o(1)}$ bits per each bidder. On the other hand, they showed that for any $r \geq 1$, there exists an r-round protocol that achieves an $O(n^{1/r+1})$ approximation by sending $O(\log n)$ bits per each bidder in each round. For the more general setting of combinatorial auctions with n subadditive bidders and m items (and hence input-size of $\exp(m)$ bits per each player), they showed that the best approximation ratio achievable by simultaneous protocols with $\text{poly}(m, n)$ communication is $\Omega(m^{1/4})$, while for any $r \geq 1$, there exists r-round protocols that achieve an approximation ratio of $\widetilde{O}(r \cdot m^{1/r+1})$. These results imply that in such markets, logarithmic rounds of interaction in the market size *suffice* to obtain an (almost) efficient allocation, i.e., a $\text{polylog}(m)$-approximation.

A natural question left open by [10] was to identify the amount of interaction *necessary* to obtain an efficient allocation in these markets. Recently, Alon, Nisan, Raz, and Weinstein [2] provided a partial answer to this question for matching markets: for any $r \geq 1$, any r-round protocol for unit-demand bidders in which each bidder sends at most $n^{o(1)}$ bits in each round can only achieve an $\Omega(n^{1/5^{r+1}})$ approximation [2]. This implies that at least $\Omega(\log\log n)$ rounds of interaction is necessary to achieve an efficient allocation in matching markets. Alon et al. [2] further conjectured that the "correct" lower bound for the convergence rate in this setting is $\Omega(\log n)$; in other words, $\Omega(\log n)$ rounds of interaction are necessary for achieving an efficient allocation.

Despite this progress for matching markets, the best known lower bounds for the more general setup of combinatorial auctions with subadditive bidders remained the aforementioned 1-round lower bound of [10], and a $(2-\varepsilon)$-approximation (for every constant $\varepsilon > 0$) for any polynomial communication protocol with unrestricted number of rounds [11]. Indeed, obtaining better lower bounds for r-round protocols was posed as an open problem by Alon et al. [2] who also mentioned that: "from a communication complexity perspective, lower bounds in this setup are more compelling, since player valuations require exponentially many bits to encode, hence interaction has the potential to reduce the overall communication from exponential to polynomial.".

1.1 Our Results and Techniques

In this paper, we resolve the aforementioned open question of Dobzinski et al. [10] and Alon et al. [2] by proving an almost tight *round-approximation* tradeoff for polynomial communication protocols in subadditive combinatorial auctions.

Main Result. *For any $r \geq 1$, any r-round protocol (deterministic or randomized) for combinatorial auctions with* subadditive *bidders that uses polynomial communication can only achieve an approximation ratio of $\Omega(\frac{1}{r} \cdot m^{1/\Theta(r)})$ to the social welfare.*

We remark that this lower bound holds even when the bidders valuations are XOS functions, a strict subclass of subadditive valuations (see Section 2.1 for definition).

Our main result, combined with the upper bound result of [10], provides a near-complete understanding of the power of each additional round in improving the quality of the allocation in subadditive combinatorial auctions. Moreover, an immediate corollary of our result is that in these markets, $\Omega(\frac{\log m}{\log\log m})$ rounds of interaction are *necessary* to achieve any efficient allocation (i.e., constant or polylogarithmic approximation), which is *tight* up to an $O(\log\log m)$ factor. The qualitative message of this theoretical result is clear: *a modest amount of interaction between individuals in a market is crucial for obtaining an efficient allocation.*

Our first step in establishing this result is proving a new lower bound for simultaneous (1-round) protocols. We deviate from [10] by considering the problem of estimating the *value* of social welfare as opposed to finding the actual allocation; this problem can only be harder in terms of proving a lower bound as any protocol that can find an approximate allocation can also be used to estimate the value of social welfare with one additional round and $O(n)$ additional communication using a trivial reduction (see Section 2.2). As a result, the kind of combinatorial arguments used in [10] seem not sufficient for our purpose and we instead prove our lower bound using information-theoretic machinery and in particular a direct-sum style argument. This counterintuitive switch to establishing a lower bound for a seemingly harder problem however leads to a more modular proof that allows us to further carry out our results to multi-round protocols.

We establish our multi-round lower bound following the multi-party round-elimination technique of Alon et al. [2]. We create a recursive family of hard distributions $\mathcal{D}_1, \mathcal{D}_2, \ldots$ whereby for any $r \geq 1$, $\mathcal{D}_r$ is the hard input distribution for r-round protocols. Each instance in $\mathcal{D}_r$ is a careful combination of *exponentially many* sub-instances sampled from $\mathcal{D}_{r-1}$. One of these sub-instances is "special" in that to solve the original instance, the players also need to solve this special sub-instance completely. On the other hand, the players are not able to identify this special sub-instance *locally* and hence need to spend one round of interaction *only* for this purpose. In other words, we prove that the first round of protocol does not convey much information about the special instance beyond its identity. Using a further round-elimination argument, we inductively show that since solving the special instance is hard for $(r-1)$-round protocols, solving the original instance should be hard for r-round protocols as well.

Similar to [2], and unlike typical two-player round-elimination arguments (see, e.g. [22, 25]), eliminating a round in our round-elimination argument requires a reduction from "low dimensional" instances (with fewer players and items) to "high dimensional" instances. This reduction is delicate as the players need to "complete" their inputs in the higher dimensional instance by *independently* sampling the "missing part" *conditioned on the first message* of the protocol *without any further communication*, while this distribution is a *correlated* distribution.

Furthermore, in contrast to [2], our sub-instances in each distribution are *overlapping* (as otherwise exponentially many sub-instances cannot be embedded inside a single polynomially larger instance) and hence may interfere with each other, potentially diminishing the role of the special instance. We overcome this obstacle by embedding these sub-instances based on a family of *small-intersection sets* to limit the potential overlap between the sub-instances and prove that solving the special instance is crucial even in the presence of these overlaps. It is worth pointing out that this approach allows us to avoid the doubly-exponential rate of growth in the size of instances across different rounds in [2], resulting in exponentially better dependence on the parameter r in our lower bound compared to [2]. Finally, since our lower bound is for estimating the *value* of social welfare (as opposed to finding an allocation), we need a different embedding argument in our reduction than the one used in [2][2]. In particular, we now embed the low dimensional instance in *multiple* places of the high dimensional instance as opposed to only one.

1.2 Other Related Work

Communication complexity of combinatorial auctions has received quite a lot of attention in the literature. It is known that for *arbitrary valuations*, exponential amount of communication is needed to obtain an $\left(m^{1/2-\varepsilon}\right)$-approximate allocation (for every constant $\varepsilon > 0$) [24] (see also [23]), and this is also tight [1, 8, 19, 21]. For *subadditive valuations*, a constant factor approximation to the social welfare can be achieved in our model using only polynomial communication [11, 12, 14–16, 20, 26] (and polynomially many rounds of interaction); in particular, Feige [15] developed a 2-approximation polynomial communication protocol for this problem and Dobzinski, Nisan, and Schapira [11] proved that obtaining $(2-\varepsilon)$-approximation (for any constant $\varepsilon > 0$) requires exponential communication (regardless of the number of rounds). Moreover, Dutting and Kesselheim [14] designed an $O(\log m)$-approximation protocol with polynomial communication for subadditive combinatorial auctions in which each bidder needs to communicate exactly once; however, this protocol still requires n rounds of interaction in our model as the players need to communicate in a round-robin fashion making the message sent by a bidder crucially depending on the messages communicated earlier by the previous bidders.

Another line of relevant research considers the case where the valuation of the bidders are chosen *independently* from a commonly known distribution (see, e.g. [17, 18]) and aims to design "simple" and simultaneous protocols that achieve an efficient allocation. The main difference between this setting and ours is that we are interested in arbitrary distributions of inputs for the bidders which are not necessarily product distributions; as already shown by the strong impossibility results of [10], the aforementioned type of protocols cannot provably exist in our model when input distributions are correalted. Finally, we point out that "incompressability" results are also known for subadditive valuations: any polynomial-length encoding of subadditive valuations must lose $\Omega(\sqrt{m})$ in precision [5, 6].

We refer the interested reader to [10] for a comprehensive summary of related work and further discussion on the role of interaction in markets.

2 PRELIMINARIES

Notation. For any integer $a \geq 1$, we let $[a] := \{1, \ldots, a\}$. We say that a set $S \subseteq [n]$ with $|S| = s$ is a *s-subset* of $[n]$. For a k-dimensional tuple $X = (X_1, \ldots, X_k)$ and index $i \in [k]$, we define $X^{<i} := (X_1, \ldots, X_{i-1})$ and $X^{-i} := (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_k)$. We use capital letters to denote random variables. For a random variable A, $\text{supp}(A)$ denotes the support of A and $\text{dist}(A)$ denotes

[2]Our problem corresponds to the problem of estimating the *size* of a maximum matching as opposed to finding an *approximate* matching in the setting of [2]. To the best of our knowledge, no non-trivial lower bounds are known for the matching size estimation problem in the setting of [2]; see [4] for further details.

its distribution. We further define $|A| := \log|\text{supp}(A)|$. We write $A \perp B \mid C$ to denote that the random variables A and B are independent conditioned on C. We use "w.p." to mean "with probability".

Intersecting families. The following combinatorial construction plays a crucial role in our proofs.

Definition 2.1. A (p, q, t, ℓ)*-intersecting family* $\mathcal{F}$ is a collection of p subsets of $[q]$ each of size t, such that for any two distinct sets $S, T \in \mathcal{F}$, $|S \cap T| \leq \ell$.

We prove the existence of an exponentially large intersecting family with a small pair-wise intersection, using a probabilistic argument. The proof is deferred to the full version of the paper [3].

Lemma 2.2. *For any integer $r \geq 1$, any parameter $\varepsilon > 0$, and any integer $k \geq \left(2e^2 \cdot r^2\right)^{\frac{1}{\varepsilon}}$, there exists a (p, q, t, ℓ)-intersecting family with $p = \exp\left(\Theta(k^{2r-2+\varepsilon})\right)$, $q = k^{2r} + r \cdot k^{2r-1}$, $t = r \cdot k^{2r-1}$, and $\ell = k^{2r-2+\varepsilon}$.*

2.1 Combinatorial Auctions

We have a set N of n bidders, and a set M of m items. Each bidder $i \in N$ has a valuation function $v_i : 2^M \to \mathbb{R}_+$, which assigns a value to each *bundle* of items (we assume $v_i(\emptyset) = 0$ and $v_i(\cdot)$ is non-decreasing). The goal is to maximize the *social welfare* defined as $\max_{(A_1,\ldots,A_n)} \sum_{i \in N} v_i(A_i)$, where $(A_1, \ldots, A_n)$ ranges over all possible allocations of items in M to bidders in N such that bidder i receives the bundle A_i.

A valuation function $v(\cdot)$ is considered *subadditive* iff for any two bundles of items $S, T \subseteq M$, $v(S \cup T) \leq v(S) + v(T)$. A valuation function is *additive* iff for any bundle $S \subseteq M$, $v(S) = \sum_{j \in S} v(\{j\})$. A valuation function is *XOS* iff there exists r additive valuation functions $a_1, \ldots, a_r$ such that for all bundles $S \subseteq M$, $v(S) = \max_r a_r(S)$. Each function a_j is called a *clause* of v and for any bundle S, any clause $a \in \arg\max_r a_r(S)$ is referred to as a *maximizing clause* of S. Finally, a valuation function is *unit-demand* iff for any $S \subseteq M$, $v(S) = \max_{j \in S} v(\{j\})$. It is easy to verify that both XOS and unit-demand functions are also subadditive.

Notice that in general, subadditive and XOS valuation functions require $\exp(m)$ many bits for representation, while unit-demand valuation functions can be represented with $O(m)$ numbers, i.e., by describing the value of each singleton set. As such, in subadditive combinatorial auctions, we are interested in protocols that can reduce the communication from exponential in m to polynomial, while in unit-demand auctions, we mainly seek protocols that reduce the communication from linear in m to logarithmic.

2.2 Communication Model

We use the (number-in-hand) multiparty communication model with shared blackboard: there are n players (corresponding to the bidders) receiving inputs $(x_1, \ldots, x_n)$, jointly distributed according to a prior distribution $\mathcal{D}$ on $\mathcal{X}_1 \times \ldots \times \mathcal{X}_n$. The communication proceeds in *rounds* whereby in each round r, the players *simultaneously* write a message on a *shared blackboard* visible to all parties. In a deterministic protocol, the message sent by any player i in each round can only depend on the private input of the player, i.e., x_i, plus the messages of all players in previous rounds, i.e., the content of the blackboard. In a randomized protocol, we further allow the players to have access to both public and private randomness and the message of players can depend on them as well.

For a protocol π, we use $\Pi = (\Pi_1, \ldots, \Pi_n)$ to denote the transcript of the message communicated by the players (i.e., the content of the blackboard). In addition to the n players, there exists also a $(n+1)$-th party called the *referee* which does not have any input, and is responsible for outputting the answer in the last round, solely based on content of the blackboard Π (plus the public randomness in case of randomized protocols). Finally, the *communication cost* of the protocol π, denoted by $\|\pi\|$, is the sum of worst-case length of the messages communicated by all players, i.e., $\|\pi\| = \sum_{i=1}^{n} |\Pi_i|$.

Approximation guarantee. We consider protocols that are required to estimate the *maximum value of social welfare* in any instance I of a combinatorial auction (denoted by $\mathrm{sw}(I)$). More formally, a δ-error α-approximation protocol needs to, for each input instance I sampled from $\mathcal{D}$, output a number in the range $[\frac{1}{\alpha} \cdot \mathrm{sw}(I), \mathrm{sw}(I)]$ w.p. at least $1 - \delta$, where the randomness is over the distribution $\mathcal{D}$ (and the randomness of protocol in case of randomized protocols).

This problem is provably easier than finding an approximate allocation in the interactive setting: any r-round protocol for finding an approximate allocation can be used to obtain an $(r + 1)$-round protocol for estimating the value of social welfare with $O(n)$ additional communication; simply compute the approximate allocation in the first r rounds and spend one additional round in which each player declares her value for the assigned bundle to the referee. It was shown very recently in [7] that this *loss of one round* in the reduction is unavoidable (see Section 3 for further details). However, this extra one round is essentially negligible for our purpose as we are interested in the asymptotic dependence of the approximation ratio and the number of rounds.

2.3 Tools from Information Theory

We briefly review some basic definitions and facts from information theory that are used in this paper. We refer the interested reader to the excellent text by Cover and Thomas [9] for an introduction to this field, and the proofs of the claims in this section (see Chapter 2).

In the following, we denote the *Shannon Entropy* of a random variable A by $\mathbb{H}(A)$ and the *mutual information* of two random variables A and B by $\mathbb{I}(A\,;B) = \mathbb{H}(A) - \mathbb{H}(A \mid B) = \mathbb{H}(B) - \mathbb{H}(B \mid A)$. The proof of the following basic properties of entropy and mutual information can be found in [9], Chapter 2.

Fact 2.3. *Let A, B, and C be three (possibly correlated) random variables.*

(1) $0 \leq \mathbb{H}(A) \leq |A|$, *and* $\mathbb{H}(A) = |A|$ *iff A is uniformly distributed over its support.*
(2) $\mathbb{I}(A\,;B \mid C) \geq 0$. *The equality holds iff A and B are* independent *conditioned on C.*
(3) $\mathbb{H}(A \mid B, C) \leq \mathbb{H}(A \mid B)$. *The equality holds iff* $A \perp C \mid B$.
(4) $\mathbb{I}(A, B\,;C) = \mathbb{I}(A\,;C) + \mathbb{I}(B\,;C \mid A)$ (chain rule of mutual information).
(5) Suppose $f(A)$ is a deterministic function of A, then $\mathbb{I}(f(A)\,;B \mid C) \leq \mathbb{I}(A\,;B \mid C)$ (data processing inequality).

We also use the following two simple propositions. The proofs are standard (see [3]).

Proposition 2.4. *For variables A, B, C, D, if $A \perp D \mid C$, then,* $\mathbb{I}(A\,;B \mid C) \leq \mathbb{I}(A\,;B \mid C, D)$.

Proposition 2.5. *For variables A, B, C, D, if $A \perp D \mid B, C$, then,* $\mathbb{I}(A\,;B \mid C) \geq \mathbb{I}(A\,;B \mid C, D)$.

For two distributions μ and ν over the same probability space, the *Kullback-Leibler divergence* between μ and ν is defined as $\mathbb{D}(\mu \mid\mid \nu) := \mathbb{E}_{a\sim\mu}\left[\log \frac{\Pr_\mu(a)}{\Pr_\nu(a)}\right]$. We have,

Fact 2.6. *For random variables A, B, C,*

$$\mathbb{I}(A\,;B \mid C) = \mathop{\mathbb{E}}_{(b,c)\sim\mathrm{dist}(B,C)}\Big[\mathbb{D}(\mathrm{dist}(A|C = c) \mid\mid \mathrm{dist}(A|B = b, C = c))\Big].$$

We denote the *total variation distance* between two distributions μ and ν over the same probability space Ω by $\|\mu - \nu\| = \frac{1}{2} \cdot \sum_{x\in\Omega} \left|\Pr_\mu(x) - \Pr_\nu(x)\right|$.

Fact 2.7 (Pinsker's inequality). *For any two distributions μ and ν,* $\|\mu - \nu\| \leq \sqrt{\frac{1}{2} \cdot \mathbb{D}(\mu \mid\mid \nu)}$.

Fact 2.8. *Suppose μ and ν are two distributions for an event $\mathcal{E}$, then,* $\Pr_\mu(\mathcal{E}) \leq \Pr_\nu(\mathcal{E}) + \|\mu - \nu\|$.

3 WARM UP: A LOWER BOUND FOR SIMULTANEOUS PROTOCOLS

Our main lower bound result is based on analyzing a recursive family of distributions. As a warm up, we analyze the base case of this recursive construction in this section and prove a lower bound for 1-round (simultaneous) protocols. Formally,

Theorem 3.1. *For any sufficiently small constant $\varepsilon > 0$, any simultaneous protocol (possibly randomized) for combinatorial auctions with subadditive (even XOS) bidders that can approximate the value of social welfare to a factor of $m^{\frac{1}{3}-\varepsilon}$ requires* $\exp\left(m^{\Omega(\varepsilon)}\right)$ *bits of communication.*

It is worth mentioning that the bound established in Theorem 3.1 on the approximation ratio of simultaneous protocols is *tight*. Previously, Dobzinski et al. [10] developed a simultaneous protocol that can approximate the social welfare up to an $\widetilde{O}(m^{1/3})$ factor using only polynomial communication. As such, Theorem 3.1 already makes a small contribution for simultaneous protocols. But more importantly, this theorem sets the stage for our main lower bound result in Section 4.

As pointed out earlier, Dobzinski et al. [10] have previously proved an $\Omega(m^{1/4})$ lower bound on the approximation ratio of the protocols that can find an approximate allocation. We should remark that this lower bound of [10] and our lower bound in Theorem 3.1 are *incomparable* in that neither imply (or strengthen) the other. The reason is that while the problem of estimating the social welfare is provably easier than the problem of finding an approximate allocation, the reduction requires one additional round of interaction and hence, in general, a simultaneous protocol for the problem of finding the allocation only implies a 2-round (and not a simultaneous) protocol for the social welfare estimation problem[3]. Interestingly, for $n = 2$ players, Braverman et al. [7] very recently showed that estimating the social welfare is indeed provably *harder* than finding an approximate allocation for *simultaneous* protocols. In the light of this result, it seems plausible that one can improve the protocol of [10] and obtain an $O(m^{1/4})$-approximation protocol for finding an approximate allocation (matching the lower bound of [10]); however, Theorem 3.1 suggests that such a protocol should necessarily be oblivious to the welfare of the allocation it provides.

3.1 A Hard Input Distribution for Simultaneous Protocols

In this section, we propose a hard input distribution $\mathcal{D}_1$ for simultaneous protocols and state several of its properties that are needed in proving the lower bound for this distribution. We start by providing an informal description of the distribution $\mathcal{D}_1$.

Let k be an integer and consider a set N of $n = k^2$ players and M of $m = k^3$ items. Each bidder $i \in N$, is given an exponentially large (in k) collection $\mathcal{F}_i$ of item-sets of size k each, such that for all $S \subseteq M$, $v_i(S) = \max_{T \in \mathcal{F}_i} |S \cap T|$ (recall that the input to player i is the valuation function $v_i(\cdot)$). Additionally, the sets in $\mathcal{F}_i$ are "barely overlapping", in the sense that for any two sets $S, T \in \mathcal{F}_i$, $|S \cap T| < k^{\varepsilon}$ (for any constant $\varepsilon > 0$).

This construction ensures that *locally* each player is confronted with exponentially many high value bundles (sets in $\mathcal{F}_i$) that look "exactly the same". However, these collections across different players are chosen in a correlated way such that except for a single "special bundle" $T_j \in \mathcal{F}_i$ (for each $i \in N$), the items in all other bundles are chosen (mostly) from a (relatively small) set of k^2 "shared" items across all players. The special bundles on the other hand consist of "unique" items.

[3]Note however that the $\widetilde{O}(m^{1/3})$-approximation protocol of [10] can already compute the welfare of the allocated allocation and hence does *not* need an additional round for estimating the social welfare, implying the tightness of the bounds in Theorem 3.1.

This imply that *globally* each player is assigned a special bundle and these special bundles are crucial to obtaining any $k^{1-\varepsilon}$-approximate allocation (recall that $k^{1-\varepsilon} = m^{\frac{1-\varepsilon}{3}}$).

We then use an additional randomization trick to ensure that any instance sampled from $\mathcal{D}_1$ either has a "large" social welfare (w.p., say, half) or a "small" one (with the remaining probability): we drop some of the bundles from the collection $\mathcal{F}_i$ of each player $i \in N$ randomly (in a correlated way), to create two sub-distributions whereby in one of them none of the special bundles are dropped and hence the social welfare is k^3, and in the other one all special bundles are dropped and hence the social welfare is at most $k^{2+\varepsilon}$ (k^2 for shared items plus k^ε intersection from any other bundle (in $\mathcal{F}_i$) for each of the k^2 players). This completes the description of our hard distribution. We now formally define $\mathcal{D}_1$.

Distribution $\mathcal{D}_1(N, M)$. A hard input distribution for simultaneous protocols.
Input: Collections N of $n = k^2$ players and M of $m = 2k^3$ items.
Output: A set of n valuation functions $(v_1, \ldots, v_n)$ for the players in N.

(1) Let $\mathcal{S} = \left\{S_1, \ldots, S_p\right\}$ be a (p, q, t, ℓ)-intersecting family with $p = \exp(\Theta(k^\varepsilon))$, $q = k^2 + k$, $t = k$, and $\ell = k^\varepsilon$ (guaranteed to exist by Lemma 2.2).
(2) Pick $j^\star \in [p]$ and $\theta \in \{0, 1\}$ independently and uniformly at random.
(3) For each player $i \in N$ independently,
  (a) Denote by $\mathcal{F}_i$ the *private collection* of player i (used below to define the valuation function v_i), initialized to be a copy of $\mathcal{S}$ on the universe $[q]$.
  (b) Let $x_i \in \{0, 1\}^p$ be a p-dimensional vector whereby $x_i(j^\star) = \theta$ and for any $j \neq j^\star$, $x_i(j)$ is chosen uniformly at random from $\{0, 1\}$.
  (c) For any $j \in [p]$, if $x_i(j) = 0$, remove the set S_j from $\mathcal{F}_i$, and otherwise keep S_j in $\mathcal{F}_i$.
(4) Pick a random permutation σ of M. For the i-th player in N, map the j-th item in $[q] \setminus S_{j^\star}$ to $\sigma(j)$. Moreover, map the j-th item in $S_{j^\star}$ to $\sigma(k^2 + (i-1) \cdot k + j)$. Under this mapping, the private collection $\mathcal{F}_i$ of player i consists of at most p sets of $t = k$ items from M.
(5) For all $i \in N$, define the valuation function of player i as $v_i(S) = \max_{T \in \mathcal{F}_i} |S \cap T|$.

We use $\mathcal{D}_1$ to denote the distribution $\mathcal{D}_1(N, M)$ whenever the sets N and M are clear from the context (or are irrelevant). We make several observations about the distribution $\mathcal{D}_1$.

Observation 3.2. *The valuation function of each bidder $i \in [n]$ in the distribution $\mathcal{D}_1$ is an XOS valuation (and hence is also subadditive) whereby each set $T \in \mathcal{F}_i$ defines a clause in which all items in T have value 1 and all other items have value 0.*

For any player $i \in N$, we define the *labeling function* ϕ_i as the function used to map the items in $[q]$ to M. Notice that ϕ_i is a function of σ and index $j^\star$.

Observation 3.3. *The input to player i can be uniquely identified by the pair (x_i, ϕ_i), as x_i defines the private collection $\mathcal{F}_i$ over the items $[q]$, and ϕ_i specifies the actual labeling of the items in M in the instance.*

We also point out a crucial property of this distribution: each player $i \in N$ is *oblivious* to which of the sets S_j (for $j \in [p]$), is the set $S_{j^\star}$. More formally,

Observation 3.4. *Conditioned on the input (x_i, ϕ_i) to player i, the index $j^\star \in [p]$ is chosen uniformly at random.*

Recall that for an instance $I \sim \mathcal{D}_1$, $\mathrm{sw}(I)$ denotes the maximum value of social welfare, i.e., $\mathrm{sw}(I) := \max_{(A_1,\ldots,A_n)} \sum_{i \in N} v_i(A_i)$, where $(A_1, \ldots, A_n)$ ranges over all possible allocation of items. The following lemma establishes a bound on the social welfare of any instance sampled from $\mathcal{D}_1$. The proof is deferred to the full version of the paper [3].

Lemma 3.5. *For any $I \sim \mathcal{D}_1$, (i) if $\theta = 1$, then $\mathrm{sw}(I) = k^3$, and (ii) if $\theta = 0$, then $\mathrm{sw}(I) \leq 2k^{2+\varepsilon}$.*

3.2 The Lower Bound for Distribution $\mathcal{D}_1$

Let π be a public coin simultaneous protocol that can output a $\left(m^{\frac{1-\varepsilon}{3}}\right)$-approximation to the social welfare of any instance $I \sim \mathcal{D}_1$, w.p. of failure $\delta \leq 1/3$. In this section, we prove that the communication cost of the protocol π needs to be at least $\exp(k^{\Omega(\varepsilon)})$ bits. Note that by (the easy direction of) Yao's minimax principle [27], we only need to consider deterministic protocols on the distribution $\mathcal{D}_1$ to prove this result.

The intuition behind the proof is as follows. By Lemma 3.5, the social welfare in the given instance changes by a factor of $k^{1-\varepsilon}$ depending on the value of θ. This implies that any $k^{1-\varepsilon} = m^{\frac{1-\varepsilon}{3}}$ approximation algorithm for the social welfare can also determine the value of θ. Using this, we can argue that the message sent by the players needs to reveal $\Omega(1)$ bit of information about the parameter θ. Roughly speaking, this means that each of the n players is responsible for revealing $\Omega(1/n)$ bit about θ in average.

Furthermore, recall that the input to player $i \in N$ can be seen as a tuple (x_i, ϕ_i) (by Observation 3.3) and that $\theta = x_i(j^\star)$. Additionally, by Observation 3.4, given input (x_i, ϕ_i) to player i, the index $j^\star$ is chosen uniformly at random from $[p]$ and hence player i is oblivious to which index of x_i corresponds to the parameter θ. This essentially means that player i needs to reveal $\Omega(p/n)$ bits about the vector x_i to be able to reveal $\Omega(1/n)$ bit about $x_i(j^\star)$, hence forcing i to communicate $\Omega(p/n) = \exp\left(k^{\Omega(\varepsilon)}\right)$ bits also. To make the latter intuition precise, we argue that while the message sent by one player can, in principle, be used to infer information about the input of another player (as the input of the players are correlated), this extra information is limited to an "easy part", containing only $(\sigma, j^\star)$ that can even be assumed to be known to referee (but not players) beforehand. This allows us to "break" the information revealed to the referee to smaller pieces sent by each player, hence arguing that each player is indeed directly responsible for communicating the information about her input. We now formalize this intuition. We first need the following notation.

Notation. We use $\Pi = (\Pi_1, \ldots, \Pi_n)$ to denote the random variable for the transcript of the messages communicated in π. For any player $i \in N$, and any $j \in [p]$, we use the random variable $X_{i,j} \in \{0, 1\}$ to denote the value of $x_i(j)$, i.e., $X_{i,j} = 1$ iff the set $S_j \in \mathcal{S}$ is included in the private collection $\mathcal{F}_i$. We further define X_i for $i \in N$ as the vector $X_i := (X_{i,1}, \ldots, X_{i,p})$. We use Σ to denote the random variable for the permutation σ, J for the index $j^\star$, and Θ for the parameter θ. For each player $i \in N$, Φ_i denotes the random variable for the labeling function ϕ_i.

Recall that (Σ, J) is the "easy part" of the input: the part that we assume the referee (but not each individual player) knows beforehand. Assuming this knowledge can only strengthen our lower bound. We start by arguing that the protocol π needs to reveal $\Omega(1)$ bits of information about the value of parameter θ in the distribution. The proof is deferred to the full version of the paper [3].

Claim 3.6. $\mathbb{I}(\Theta ; \Pi \mid \Sigma, J) = \Omega(1)$.

We now show that the information revealed about Θ by the message Π is at most the sum of information revealed by each message Π_i for $i \in N$ individually. In other words, one does not gain an extra information by combining the messages of players (after conditioning on what is revealed by (Σ, J) already).

Claim 3.7. $\mathbb{I}(\Theta\,;\Pi \mid \Sigma, J) \leq \sum_{i \in N} \mathbb{I}(\Theta\,;\Pi_i \mid \Sigma, J)$.

Proof. We have,

$$\mathbb{I}(\Theta\,;\Pi \mid \Sigma, J) = \sum_{i \in N} \mathbb{I}(\Theta\,;\Pi_i \mid \Pi^{<i}, \Sigma, J) \leq \sum_{i \in N} \mathbb{I}(\Theta\,;\Pi_i \mid \Sigma, J)$$

where the equality is by chain rule (Fact 2.3-(4)), and the inequality follows from Proposition 2.5, as we show below that $\Pi_i \perp \Pi^{<i} \mid \Theta, \Sigma, J$, or equivalently $\mathbb{I}(\Pi_i\,;\Pi^{<i} \mid \Theta, \Sigma, J) = 0$ (by Fact 2.3-(2)).

As stated in Observation 3.3, the input of player $i \in N$ is uniquely determined by (x_i, ϕ_i) and hence Π_i is a deterministic function of variables X_i and Φ_i. Moreover, Φ_i is also uniquely determined by (Σ, J), hence, conditioned on (Σ, J), Π_i is only a function of X_i. On the other hand, conditioned on (Θ, Σ, J), X_i and $X^{<i}$ are chosen independently of each other in the distribution $\mathcal{D}_1$ (as $X_{i,j^\star} = \theta$ and the rest of X_i is chosen uniformly at random from $\{0, 1\}$). This implies that $\mathbb{I}(X_i\,;X^{<i} \mid \Theta, \Sigma, J) = 0$. As stated earlier, Π_i is a function of X_i and $\Pi^{<i}$ is a function of $X^{<i}$ alone (conditioned on (Θ, Σ, J)), hence, by data processing inequality (Fact 2.3-(5)), $\mathbb{I}(\Pi_i\,;\Pi^{<i} \mid \Theta, \Sigma, J) = 0$ as well. □

We now use a direct-sum style argument to prove that if a player $i \in N$ wants to communicate c bits about θ, she needs to communicate (essentially) $p \cdot c$ bits about her input.

Lemma 3.8. *For any $i \in N$, $\mathbb{I}(\Theta\,;\Pi_i \mid \Sigma, J) \leq |\Pi_i| / p$.*

Proof. We have,

$$\mathbb{I}(\Theta\,;\Pi_i \mid \Sigma, J) = \mathop{\mathbb{E}}_{j \in [p]} \Big[\mathbb{I}(\Theta\,;\Pi_i \mid \Sigma, J = j) \Big] = \mathop{\mathbb{E}}_{j \in [p]} \Big[\mathbb{I}(X_{i,j}\,;\Pi_i \mid \Sigma, J = j) \Big]$$

$$(\Theta = X_{i,j} \text{ conditioned on } J = j)$$

$$= \frac{1}{p} \cdot \sum_{j \in [p]} \mathbb{I}(X_{i,j}\,;\Pi_i \mid \Sigma, J = j)$$

$$(\text{the index } j^\star \text{ is chosen uniformly at random from } [p])$$

Define Σ^{-i} as the part of permutation Σ that does not affect the labeling function Φ_i of player i, i.e., the values of $\sigma(k^2 + 1) \ldots \sigma(k^2 + (i-1) \cdot k)$ and $\sigma(k^2 + i \cdot k + 1), \ldots, \sigma(k^3)$. With this notation, Σ can be written as a function of Φ_i, Σ^{-i}, and J (as J and Φ_i uniquely define the rest of Σ outside Σ^{-i}). Consequently, we can write,

$$\mathbb{I}(\Theta\,;\Pi_i \mid \Sigma, J) = \frac{1}{p} \cdot \sum_{j \in [p]} \mathbb{I}(X_{i,j}\,;\Pi_i \mid \Sigma^{-i}, \Phi_i, J = j)$$

Our goal is now to drop the conditioning on the event "$J = j$". To do so, notice that the distribution of (Σ^{-i}, Φ_i) is independent of the event $J = j$; this is immediate to see as Σ^{-i} is independent of Φ_i and $J = j$, and Φ_i is independent of $J = j$ by Observation 3.4. Moreover, $X_{i,j}$ is independent of all $(\Sigma^{-i}, \Phi_i, J = j)$ (as it is uniform over $\{0, 1\}$) and furthermore, Π_i is a function of Φ_i, X_i, which are independent of $J = j$. Consequently, we can drop the conditioning in the above information term and obtain that,

$$\mathbb{I}(\Theta\,;\Pi_i \mid \Sigma, J) = \frac{1}{p} \cdot \sum_{j \in [p]} \mathbb{I}(X_{i,j}\,;\Pi_i \mid \Sigma^{-i}, \Phi_i) \leq \frac{1}{p} \cdot \sum_{j \in [p]} \mathbb{I}(X_{i,j}\,;\Pi_i \mid X_i^{<j}, \Sigma^{-i}, \Phi_i)$$

$$(\text{by Proposition 2.4 as } X_{i,j} \perp X_i^{<j} \mid \Sigma^{-i}, \Phi_i)$$

$$= \frac{1}{p} \cdot \mathbb{I}(X_i\,;\Pi_i \mid \Sigma^{-i}, \Phi_i) \leq \frac{1}{p} \cdot \mathbb{H}(\Pi_i \mid \Sigma^{-i}, \Phi_i) \leq \frac{1}{p} \cdot \mathbb{H}(\Pi_i) \leq \frac{1}{p} \cdot |\Pi_i|$$

where the equality in the second line is by chain rule (Fact 2.3-(4)), and inequalities are by Fact 2.3-(1) and Fact 2.3-(3). ◻

We can now conclude the following lemma.

Lemma 3.9. *Communication cost of π is $\Omega(p)$.*

Proof. $\|\pi\| = \sum_{i \in N} |\Pi_i| \geq p \cdot \sum_{i \in N} \mathbb{I}(\Theta\,; \Pi_i \mid \Sigma, J) \geq p \cdot \mathbb{I}(\Theta\,; \Pi \mid \Sigma, J) = \Omega(p)$. where the last three equations are by, respectively, Lemma 3.8, Claim 3.7, and Claim 3.6. ◻

Theorem 3.1 now follows from Lemma 3.9 by re-parameterizing ε above by some $\Theta(\varepsilon)$ and noting that $p = \exp(\Theta(k^{\varepsilon})) = \exp\left(m^{\Omega(\varepsilon)}\right)$ (as $m = k^3$).

4 MAIN RESULT: A LOWER BOUND FOR MULTI-ROUND PROTOCOLS

In this section, we establish our main result. Formally,

Theorem 4.1. *For any integer $1 \leq r \leq o\left(\frac{\log m}{\log\log m}\right)$, and any sufficiently small constant $\varepsilon > 0$, any r-round protocol (possibly randomized) for combinatorial auctions with subadditive (even XOS) bidders that can approximate the value of social welfare to a factor of $\left(\frac{1}{r} \cdot m^{\frac{1-\varepsilon}{2r+1}}\right)$ requires $\exp\left(m^{\Omega(\frac{\varepsilon}{r})}\right)$ bits of communication.*

We start by introducing the recursive family of hard input distributions that we use proving in Theorem 4.1 and then establish a lower bound for this distribution.

4.1 A Hard Input Distribution for r-Round Protocols

Our hard distribution $\mathcal{D}_r$ for r-round protocols is defined recursively with its base case ($r = 1$ case) being the distribution $\mathcal{D}_1$ introduced in Section 3.1. We first give an informal description of $\mathcal{D}_r$.

Let k be an integer and consider a set N of $n_r = k^{2r}$ players and a set M of $m_r = (r+1) \cdot k^{2r+1}$ items. The players are partitioned (arbitrary) between k^2 groups $N_1, \ldots, N_{k^2}$ each of size n_{r-1}. Fix a group N_g and for any player $i \in N_g$, we create an exponentially large (in k) collection C_i of item-sets of size m_{r-1} (over the universe M), such that the for any two sets $S, T \in C_i$, $|S \cap T| \leq k^{2r-2+\varepsilon}$ (for any constant $\varepsilon > 0$).

The *local* view of player $i \in N_g$ is as follows: over each set $S_j \in C_i$, we create an $(r-1)$-round instance of the problem, namely instance $I_{i,j}$, sampled from the distribution $\mathcal{D}_{r-1}$ with the set of players being N_g and the set of items being S_j, and then let the input of player i be the collective input of the i-th player in all these instances. In other words, player i finds herself "playing" in exponentially many "$(r-1)$-round instances" of $\mathcal{D}_{r-1}$.

On the *group level*, the input to players inside a group N_g are highly correlated: for each player $i \in N_g$, one of the instances, namely $I_{i,j^\star}$, is an "special instance" in the sense that *all* players in the group N_g has a "consistent" view of this instance, i.e., the collective view of players $1, \ldots, n_{r-1}$ in N_g on the instances $I_{1,j^\star}, \ldots, I_{n_{r-1},j^\star}$ forms a valid instance sampled from $\mathcal{D}_{r-1}$. However, for any other index $j \neq j^\star$, the collective view of players in N_g in the instances $I_{1,j^\star}, \ldots, I_{n_{r-1},j^\star}$ forms a "pseudo instance" that is *not* sampled from $\mathcal{D}_{r-1}$; these pseudo instances are created by sampling the input of each player *independently* according to $\mathcal{D}_{r-1}$. Note however that while the pseudo instances and the special instance of a player are fundamentally different, each player is oblivious to this difference, i.e., which instance is the special instance.

Finally, the input to players across the groups, i.e., the *global* input, is further correlated: the set of items in the special instances of players in a group N_g is a "unique" set of items (across all groups), while *all* other instances, across all groups, are constructed over a set of k^{2r} "shared" items. This correlation makes the special instance of a player i, in some sense, the *only important* instance: to obtain a large allocation, the players need to ultimately solve the problem for these special instances.

We now formally define distribution $\mathcal{D}_r$. In the following, for simplicity of exposition, we assume that the distribution $\mathcal{D}_r$, in addition to the valuation function of players, also outputs the *private collections* (defined similarly as in $\mathcal{D}_1$) of players that are used to define these functions[4].

Distribution $\mathcal{D}_r(N, M)$. A hard input distribution for r-round protocols (for $r \geq 2$).
Input: Collections N of $n_r = k^{2r}$ players and M of $m_r = (r+1) \cdot k^{2r+1}$ items.
Output: A set of n_r valuation functions $(v_1, \ldots, v_{n_r})$ for the players in N and n_r private collections $(\mathcal{F}_1, \ldots, \mathcal{F}_{n_r})$ used to define the valuation functions.

---

(1) Let $\mathcal{S}_r = \left\{S_1, \ldots, S_p\right\}$ be a (p_r, q_r, t_r, ℓ_r)-intersecting family with parameters $p_r = p = \exp(\Theta(k^{\varepsilon}))$, $q_r = k^{2r} + r \cdot k^{2r-1}$, $t_r = r \cdot k^{2r-1}$, and $\ell_r = k^{2r-2+\varepsilon}$ (guaranteed to exist by Lemma 2.2 as $k = m^{\Omega(1/r)} = \omega(r^{2/\varepsilon})$ by the assumption that $r = o\left(\frac{\log m}{\log\log m}\right)$).
(2) Arbitrary group the players into k^2 groups $\mathcal{N} = (N_1, \ldots, N_{k^2})$, whereby each group contains exactly $n_{r-1} = k^{2r-2}$ players.
(3) Pick an index $j^\star \in [p]$ uniformly at random and sample an instance $I_r^\star \sim \mathcal{D}_{r-1}([n_{r-1}], S_{j^\star})$.
(4) For each group $N_g \in \mathcal{N}$ independently,
  (a) Define $I_{N_g}^\star$ as $I_r^\star$ by mapping the players in $[n_{r-1}]$ to N_g.
  (b) For each player $i \in N_g$ *independently*, create p instances $I^{(i)} := (I_{i,1}, \ldots, I_{i,p})$ whereby for all $j \neq j^\star$, $I_{i,j} \sim \mathcal{D}_{r-1}(N_g, S_j)$, and $I_{i,j^\star} = I_{N_g}^\star$.
  (c) For a player $i \in N_g$ and index $j \in [p]$, let $\mathcal{F}_{i,j}$ be the set of *private collection* of that player in instance $I_{i,j}$ and let $\mathcal{F}_i = \bigcup_{j \in [p]} \mathcal{F}_{i,j}$.
(5) Pick a random permutation σ of M. For each $g \in [k^2]$ and group N_g, map the k^{2r} items in $[q_r] \setminus S_{j^\star}$ to $\sigma(1), \ldots, \sigma(k^{2r})$, and the t_r items in $S_{j^\star}$ to $\sigma((g-1) \cdot t_r + 1) \ldots \sigma(g \cdot t_r)$ (and for each player $i \in N_g$, update the item set of $\mathcal{F}_i$ and underlying instances $I_{i,1}, \ldots, I_{i,p}$ accordingly).
(6) For any player $i \in N$, define the valuation function of player i as $v_i(S) = \max_{T \in \mathcal{F}_i} |S \cap T|$ (note that these valuation functions are XOS valuation; see Observation 3.2).

We make several observations about the distribution $\mathcal{D}_r$. Recall that $\mathcal{F}_i$ denotes the private collection of player $i \in N$ that is used to define the valuation function v_i. By construction, the size of the sets inside each private collection is equal across any two distributions $\mathcal{D}_r$ and $\mathcal{D}_{r'}$ and hence is equal to k (by definition of distribution $\mathcal{D}_1$). A simple property of these sets is that,

Observation 4.2. *For any player $i \in N$, and any set $T \in \mathcal{F}_i$, the set T is chosen uniformly at random from all k-subsets of M.*

Fix any group $N_g \in \mathcal{N}$ and any player $i \in N_g$. The input to player i can be seen as the "view" of i in the p instances $I^{(i)} := (I_{i,1}, \ldots, I_{i,p})$, i.e., the input of the i-th player (in N_g) in $I_{i,j}$ (for all $j \in [p]$) and *not* the whole instance. However, in the following, we slightly abuse the notation and use $I_{i,j}$ to also denote the view of player i in the instance $I_{i,j}$. Moreover, we point out that $I_{i,j}$ is defined

[4]Strictly speaking, this is a redundant information as the valuation functions can uniquely determine the private collections; however, we include this redundant output for the ease of presentation.

over the set of items S_j; hence, the complete input to player i is the pair $(I^{(i)}, \phi_i)$ where ϕ_i is the labeling function to map the items in S_j to M (see also Observation 3.3).

For any player $i \in N$, we refer to the instance $I_{i,j^\star}$ of player i as the *special instance* of player i, and to all other instances $I_{i,j}$ for $j \neq j^\star$ as *fooling instances.*

Observation 4.3. *For any group $N_g \in \mathcal{N}$, the joint input of all players $i \in N_g$ in their special instances $I_{i,j^\star}$ form the instance $I^\star_{N_g}$ that is sampled from the distribution $\mathcal{D}_{r-1}$.*

On the other hand, the fooling instances of players $i \in N_g$ are sampled *independently* and hence the joint distribution of the players on their instances $I_{i,j}$ is *not* sampled from $\mathcal{D}_{r-1}$. Nevertheless, this difference is *not evident* to the player i.

Observation 4.4. *For any player $i \in N$, conditioned on the input $(I^{(i)}, \phi_i)$ given to the player i, the index $j^\star$ is chosen uniformly at random from $[p]$.*

Additionally,

Observation 4.5. *The distribution of collection of instances $\mathcal{I} := (I^{(1)}, \ldots, I^{(n_r)}) \sim \mathcal{D}_r \mid I^\star_r, \sigma, j^\star$ is a* product *distribution as instances in Line (4b) are sampled independently (except for instances $I_{i,j^\star} = I^\star_r$ which are already conditioned on above).*

Another important property of the special instances in distribution $\mathcal{D}_r$ is that,

Observation 4.6. *The special instances $I^\star_{N_1}, \ldots, I^\star_{N_{k^2}}$ are supported on disjoint set of items (according to the mapping σ).*

Notice that we can trace the special instances into a *unique* path $I^\star_r \to I^\star_{r-1} \to \ldots \to I^\star_2$, whereby $I^\star_2$ is sampled from the distribution $\mathcal{D}_1$. We use $\theta^\star$ to denote the parameter θ (in $\mathcal{D}_1$) in the instance $I^\star_2$ in this path. The following lemma proves a key relation between $\theta^\star$ and social welfare of the sampled instance. The proof is technical and is deferred to the full version of the paper [3].

Lemma 4.7. *For any instance $I \sim \mathcal{D}_r$:*

$$\Pr\left(\text{sw}(I) \geq k^{2r+1} \mid \theta^\star = 1\right) = 1 \tag{1}$$

$$\Pr\left(\text{sw}(I) \leq 2r \cdot k^{2r+2\varepsilon} \mid \theta^\star = 0\right) = 1 - r \cdot \exp\left(-\Omega(k^\varepsilon)\right) \tag{2}$$

4.2 The Lower Bound for Distribution $\mathcal{D}_r$

Let π be a r-round protocol that can output a $\left(\frac{1}{r} \cdot m_r^{\frac{1-2\varepsilon}{2r+1}}\right)$-approximation to the social welfare of any instance $I \sim \mathcal{D}_r$, w.p. of failure $\delta < 1/4$. In this section, we prove that the communication cost of the protocol π needs to be at least $\exp(\Omega(k^\varepsilon))$ bits. By (the easy direction of) Yao's minimax principle [27], it suffices to prove this lower bound for deterministic algorithms.

We start by providing a detailed overview of the proof. First, by Lemma 4.7 we can argue that the protocol π is also a $(\delta + o(1))$-error protocol for estimating the parameter $\theta^\star$, and hence we prove the lower bound for $\theta^\star$-estimation problem instead. Recall that in any instance $I_r \sim \mathcal{D}_r$, the value of $\theta^\star$ is equal to the value of $\theta^\star$ in the underlying special instance $I^\star_r$ in I_r, and that $I^\star_r$ is sampled from the distribution $\mathcal{D}_{r-1}$. Hence to "solve" the instance $I_r \sim \mathcal{D}_r$, the players need to be able to solve the instance $I^\star_r \sim \mathcal{D}_{r-1}$ as well. This suggests an inductive approach to prove the lower bound for the distribution $\mathcal{D}_r$.

Consider the first message $\Pi_1 = (\Pi_{1,1}, \ldots, \Pi_{1,n_r})$ of π. Recall that the input to any player $i \in N$ consists of p different instances (of $\mathcal{D}_{r-1}$), one of which being the instance $I^\star_r$. By Observation 4.4, each player i is oblivious to the identity of $I^\star_r$ and hence, intuitively, the message $\Pi_{1,i}$ cannot reveal more than $\approx |\Pi_{1,i}|/p$ bits of information about the instance $I_r^\star$. Considering the simultaneity of the protocol π, we can use a similar argument as in the previous section and prove that if $|\Pi_1| = o(p)$, then at most $o(1)$ bits of information is revealed about $I_r^\star$.

Now consider the second round of the protocol π. The task of players in each group $N_g \in \mathcal{N}$ is now to solve the instance $I_r^\star$ (on a separate set of players and items). As argued above, the first message of players can only reveal $o(1)$ bits of information about $I_r^\star$ and hence distribution of $I_r^\star$ is still "very close" to its original distribution $\mathcal{D}_{r-1}$, *even conditioned on the first message of players*. But $\mathcal{D}_{r-1}$ is assumed inductively to be a hard input distribution for $(r-1)$-round protocols and as π needs to solve $I_r^\star$ in $(r-1)$ rounds now, we may argue that it needs an exponential communication.

To make this intuition precise, we employ a *round-elimination* argument: Given any hard instance $I_{r-1} \sim \mathcal{D}_{r-1}$, we "embed" I_{r-1} in an r-round instance I_r sampled from $\mathcal{D}_r$ *conditioned on the first message* Π_1 of π with *no communication* between the players and then use π from the second round onwards to solve I_{r-1}. However, notice that as the number of players (and items) vary between I_r and I_{r-1}, we cannot directly apply π on I_{r-1}. Instead, the players first sample a message Π_1 (of π) according to the distribution $\mathcal{D}_r$ using public randomness. Next, each player $i \in [n_{r-1}]$ in the instance I_{r-1} mimics the role of k^2 different players (one "copy" in each group in $\mathcal{N}$ in I_r) by letting the input of each copy in the special instance (of I_r) be her input in I_{r-1} and then "completes" the rest of her input (i.e., her fooling instances in I_r) *independently* of other players to obtain an instance $I_r \sim \mathcal{D}_r \mid I_r^\star = I_{r-1}, \Pi_1$. Note that a-priori it is not clear that why such an embedding is possible since the first message Π_1 correlates the input of players in fooling instances, making independent sampling of these instances impossible. However, we show that by further conditioning on some "easy part" of the input in the first round, i.e., σ and $j^\star$ (by sampling these parts publicly also), the players can indeed implement this embedding *without any communication* and hence obtain a valid $(r-1)$-round protocol for I_{r-1}. We are now ready to present the formal proof. To continue, we need the following notation.

Notation. For any $j \in [r]$, we use $\Pi_j = (\Pi_{j,1}, \ldots, \Pi_{j,n_r})$ to denote the random variable for the transcript of the messages communicated in the round j of π. For any player $i \in N$, and any $j \in [p]$, we override the notation and use $I_{i,j}$ to also denote the random variable for the instance $I_{i,j}$ sampled in $\mathcal{D}_r$ (similarly for $I_r^\star$ and $I^{(i)}$). We further use Σ to denote the random variable for the permutation σ and J for the index $j^\star$. We start by the following simple claim.

Claim 4.8. *Protocol π can also determine the value of $\theta^\star$ w.p. $1 - \delta - o(1)$.*

Proof. By Lemma 4.7, the ratio of $\mathrm{sw}(I)$ depending on the parameter $\theta^\star$ is (w.p. $1 - o(1)$):

$$\frac{k^{2r+1}}{2r \cdot k^{2r+2\varepsilon}} = \frac{k^{1-2\varepsilon}}{2r} = \frac{m_r^{\frac{1-2\varepsilon}{2r+1}}}{2r \cdot (r+1)^{\frac{1}{2r+1}}} > \frac{1}{r} \cdot m_r^{\frac{1-2\varepsilon}{2r+1}}$$

Hence, the δ-error $\left(\frac{1}{r} \cdot m_r^{\frac{1-2\varepsilon}{2r+1}}\right)$-approximation protocol π correctly determines the value of $\theta^\star$ w.p. $1 - \delta - o(1)$. ◻

We show that as long as the first message sent by the players is not too large, this message cannot reveal much information about the special instance $I_r^\star$ embedded in the distribution $\mathcal{D}_r$. This argument is a similar to the one in Section 3.2 and we defer it to the full version of the paper [3].

Lemma 4.9. *If $|\Pi_1| = o(p/r^4)$, then $\mathbb{I}(I_r^\star \,; \Pi_1 \mid \Sigma, J) = o(1/r^4)$.*

Recall that $I_r^\star$ is the special instance in distribution $\mathcal{D}_r$ which was sampled from distribution $\mathcal{D}_{r-1}$. We define ψ_r as the distribution of $I_r^\star$ conditioned on (Π_1, Σ, J), i.e., after seeing the first message of π and the easy part of the input (Σ, J). As a corollary of Lemma 4.9, we have that this further conditioning does not change the distribution of $I_r^\star$ by much.

Claim 4.10. *If* $|\Pi_r| = o(p/r^4)$, *then*, $\mathbb{E}_{(\Pi_1,\Sigma,J)}\left[\|\psi_r - \mathcal{D}_{r-1}\|\right] = o(1/r^2)$.

Proof. We have,

$$\begin{aligned}
\mathbb{E}_{(\Pi_1,\Sigma,J)}\left[\|\psi_r - \mathcal{D}_{r-1}\|\right] &= \mathbb{E}_{(\Pi_1,\Sigma,J)}\left[\|\psi_r - \mathrm{dist}(I_r^\star \mid (\Sigma,J))\|\right] && (\mathrm{dist}(I_r^\star) = \mathcal{D}_{r-1} \text{ and } I_r^\star \perp \Sigma, J)\\
&\leq \mathbb{E}_{(\Pi_1,\Sigma,J)}\left[\sqrt{\frac{1}{2}\cdot \mathbb{D}(\psi_r \mid\mid \mathrm{dist}(I_r^\star \mid (\Sigma,J)))}\right] && \text{(by Pinsker's inequality (Fact 2.7))}\\
&\leq \sqrt{\frac{1}{2}\cdot \mathbb{E}_{(\Pi_1,\Sigma,J)}\left[\mathbb{D}(\psi_r \mid\mid \mathrm{dist}(I_r^\star \mid (\Sigma,J)))\right]} && \text{(by concavity of } \sqrt{\cdot} \text{ and Jensen's inequality)}\\
&= \sqrt{\frac{1}{2}\cdot \mathbb{I}(I_r^\star\,; \Pi_1 \mid \Sigma, J)} && \text{(by Fact 2.6)}
\end{aligned}$$

which is $o(1/r^2)$ by Lemma 4.9. $\square$

We are now ready to state the main result of this section. Define the recursive function $e(r) := e(r-1) + o(1/r^2)$ (with $e(0) = 0$). Note that $e(r) = \sum_{i=1}^{r} o(1/i^2) = o(1)$. We have,

Lemma 4.11. *For any $r \geq 1$, any r-round protocol π for determining $\theta^\star$ on $\mathcal{D}_r$ with error probability at most $\delta = 1/3 - e(r)$ requires $\Omega(p/r^4)$ communication.*

Proof. We prove this lemma inductively. The base case for $r = 1$ follows from Lemma 3.9. Now suppose the result holds for all integers smaller than r and we aim to prove it for the case of r-round protocols. Let π be a δ-error protocol for estimating $\theta^\star$ with $\delta = 1/3 - e(r)$ and assume by contradiction that the communication cost of π is $o(p/r^4)$; we use π to design a randomized $(r-1)$-round protocol π' that has communication cost $o(p/r^4)$, and errs w.p. at most $1/3 - e(r-1)$ on $\mathcal{D}_{r-1}$, and then use averaging argument to fix its randomness to obtain a deterministic protocol that contradicts the induction hypothesis.

Protocol π': An $(r-1)$-round protocol for solving instances of $\mathcal{D}_{r-1}$ using protocol π.
Input: An instance $I \sim \mathcal{D}_{r-1}$. **Output:** The value of $\theta^\star$ in I.

---

(1) Let $N = [n_r]$ and $M = [m_r]$.
(2) Using *public randomness*, the players sample $(\Pi_1, \sigma, j^\star) \sim \mathcal{D}_r(N, M)$, i.e., they sample from the joint distribution of the first message of π (denoted by Π_1), the permutation σ over M, and the index $j^\star \in [p]$.
(3) The players partition N into k^2 equal-size groups $\mathcal{N} = (N_1, \ldots, N_{k^2})$ (as is done in $\mathcal{D}_r$) and the i-th player (denoted by P_i) in I mimics the role of the i-th player in each group $N_g \in \mathcal{N}$ (denoted by $P_{i,g}$) individually, as follows:
   (a) P_i sets the input for $P_{i,g}$ (for $g \in [k^2]$) in the instance $I_{i,j^\star}$ (in $\mathcal{D}_r$) as the input of P_i in the input instance I mapped via σ to M (using the same procedure as in $\mathcal{D}_r$).

(b) P_i samples the input for $P_{i,g}$ (for $g \in [k^2]$) in all other instances $I_{i,j}$ (for $j \neq j^\star$), using *private randomness* from the distribution $I_{i,j} \sim \mathcal{D}_r \mid (I^\star = I, \Pi_1, \sigma, j^\star)$ (we prove this is indeed possible by Proposition 4.12 below).

(4) The players run the protocol π on the new sampled instance conditioned on the first message being Π_1, (i.e., run π from the second round assuming Π_1 is the content of blackboard after the first round) and output the same answer as π.

We start by arguing that π' is indeed a valid protocol; in particular, Line (3b) can be implemented without any communication. We first need some new notation. For any player $i \in N$, define $I^\star_r(i)$ as the input of player i in the instance $I_{i,j^\star} = I^\star_r$ (conditioned on Σ, J), and define $I^\star_r(-i)$ as the input of all other players in $I^\star_r$. To prove that π' is valid, it suffices to prove the following proposition.

Proposition 4.12. *The distribution* $\mathcal{I} := (I^{(1)}, \ldots, I^{(n)}) \sim (\mathcal{D}_r \mid I^\star_r, \Pi_1, \Sigma, J)$ *is a* product distribution *whereby each* $\mathcal{I}_i = I^{(i)}$ *is sampled from* $\mathcal{D}_r \mid I^\star_r(i), \Pi_1, \Sigma, J$.

Proof. For any $i \in N$, we prove that $\mathbb{I}(\mathcal{I}_i\,; \mathcal{I}^{-i}, I^\star_r(-i) \mid I^\star_r(i), \Pi_1, \Sigma, J) = 0$. By Fact 2.3-(2), this implies that $\mathcal{I}_i \perp (\mathcal{I}^{-i}, I^\star_r(-i)) \mid I^\star_r(i), \Pi_1, \Sigma, J$, hence proving the proposition. We have,

$$\begin{aligned}\mathbb{I}(\mathcal{I}_i\,; \mathcal{I}^{-i}, I^\star_r(-i) \mid I^\star_r(i), \Pi_1, \Sigma, J) &= \mathbb{I}(\mathcal{I}_i\,; \mathcal{I}^{-i}, I^\star_r(-i) \mid I^\star_r(i), \Pi_{1,i}, \Pi_1^{-i}, \Sigma, J) \quad (\text{as } \Pi_1 = \Pi_{1,i}, \Pi_1^{-i})\\ &\leq \mathbb{I}(\mathcal{I}_i\,; \mathcal{I}^{-i}, I^\star_r(-i) \mid I^\star_r(i), \Pi_{1,i}, \Sigma, J)\end{aligned}$$

since $\mathcal{I}_i \perp \Pi_1^{-i} \mid (\mathcal{I}^{-i}, I^\star_r = (I^\star_r(i), I^\star_r(-i)), \Pi_{1,i}, \Sigma, J)$ as Π_1^{-i} is a deterministic function of $\mathcal{I}^{-i}, I^\star_r, \Sigma, J$, and hence we can apply Proposition 2.5.

Furthermore,

$$\begin{aligned}\mathbb{I}(\mathcal{I}_i\,; \mathcal{I}^{-i}, I^\star_r(-i) \mid I^\star_r(i), \Pi_1, \Sigma, J) &\leq \mathbb{I}(\mathcal{I}_i\,; \mathcal{I}^{-i}, I^\star_r(-i) \mid I^\star_r(i), \Pi_{1,i}, \Sigma, J)\\ &\leq \mathbb{I}(\mathcal{I}_i\,; \mathcal{I}^{-i}, I^\star_r(-i) \mid I^\star_r(i), \Sigma, J)\end{aligned}$$

since $\left(\mathcal{I}^{-i}, I^\star_r(-i)\right) \perp \Pi_{1,i} \mid \mathcal{I}_i, I^\star_r(i), \Sigma, J)$ as $\Pi_{1,i}$ is a deterministic function of $I^{(i)} = (\mathcal{I}_i, I^\star_r(i)), \Sigma, J$ and hence we can again apply Proposition 2.5. Finally, $\mathbb{I}(\mathcal{I}_i\,; \mathcal{I}^{-i}, I^\star_r(-i) \mid I^\star_r(i), \Sigma, J) = 0$ by Observation 4.5 and Fact 2.3-(2), implying that $\mathbb{I}(\mathcal{I}_i\,; \mathcal{I}^{-i}, I^\star_r(-i) \mid I^\star_r(i), \Pi_1, \Sigma, J) = 0$ as well, proving the proposition. □

It is now easy to see that π' is indeed an $(r-1)$-round protocol: to sample from the distribution $\mathcal{D}_r \mid (I^\star = I, \Pi_1, \Sigma, J)$ in Line (3b), each player $i \in N$ needs to sample from the distribution $\mathcal{D}_r \mid (I^\star_r(i), \Pi_1, \Sigma, J)$ (by Proposition 4.12), and this is possible since $(I^\star_r(i), \Pi_1, \Sigma, J)$ are all known to i. Hence, the players do not need any communication for simulating the first round of protocol π. We now prove that.

Claim 4.13. *π' is a δ'-error protocol for $\mathcal{D}_{r-1}$ for $\delta' = 1/3 - e(r-1)$.*

Proof Sketch. Note that our goal is to calculate the probability that π' errs given an instance $I \sim \mathcal{D}_{r-1}$. For the sake of analysis, suppose that $I \sim \psi_r$ instead, i.e., is sampled from the distribution $\text{dist}(I^\star_r \mid \Pi_1, \Sigma, J)$ (according to distribution $\mathcal{D}_r$). In this case, one can see that the distribution of the r-round instance constructed by π' matches the distribution $\mathcal{D}_r$. Since π' outputs the same answer as π on this new sampled instance, and since $I = I^\star_r$ in the new instance, the probability that π' errs on ψ_r is equal to the probability that π errs on $\mathcal{D}_r$ which in turn is equal to $1/3 - e(r)$. Now notice that by Claim 4.10, the total variation distance between ψ_r and $\mathcal{D}_{r-1}$ is $o(1/r^2)$ and hence by Fact 2.8, $\Pr_{\mathcal{D}_{r-1}}(\pi' \text{ errs}) \leq \Pr_{\psi_r}(\pi' \text{ errs}) + o(1/r^2) = 1/3 - e(r-1)$. The complete proof can be found in the full version of the paper [3]. □

Lemma 4.11 now follows from Claim 4.13 by an averaging argument since we can fix the randomness in π' to obtain a deterministic protocol π'' that uses $o(p/r^4)$ bits of communication and errs w.p. at most $1/3 - e(r-1)$ on $\mathcal{D}_{r-1}$, a contradiction with the induction hypothesis. □

Theorem 4.1 now easily follows from Lemma 4.11.

Proof of Theorem 4.1. Let π be a $\left(\frac{1}{r} \cdot m^{\frac{1-2\varepsilon}{2r+1}}\right)$-approximation, $(1/4)$-error protocol for subadditive combinatorial auctions on the distribution $\mathcal{D}_r$. By Claim 4.8, π is also a $(1/4 + o(1))$-error protocol for $\theta^\star$ estimation on $\mathcal{D}_r$. Since $(1/4 + o(1)) < 1/3 - e(r)$, by Lemma 4.11, we have $\|\pi\| = \Omega(p/r^4) = \exp\left(\Theta(k^\varepsilon)\right)/r^4 = \exp\left(m^{\Omega(\varepsilon/r)}\right)/r^4 = \exp\left(m^{\Omega(\varepsilon/r)}\right)$, as $k = m^{\Omega(1/r)}$ and $r = o(\frac{\log m}{\log\log m})$. Re-parametrizing ε by $\varepsilon/2$ in the lower bound argument finalizes the proof. □

5 CONCLUSION

In this paper, we studied the role of interaction in obtaining efficient allocations in subadditive combinatorial auctions. We showed that for any $r \geq 1$, any r-round protocol that uses polynomial communication can only achieve an $\Omega(\frac{1}{r} \cdot m^{1/2r+1})$ approximation to the social welfare. This settles an open question posed by [10] and [2] on the round-approximation tradeoff of polynomial communication protocols in subadditive combinatorial auctions.

An immediate corollary of our main result is that $\Omega(\frac{\log m}{\log\log m})$ rounds of interaction are necessary for obtaining an efficient allocation in subadditive combinatorial auctions. The qualitative message of this theoretical result is that *a modest amount of interaction between individuals in a market is crucial for obtaining an efficient allocation*. This further support the point of view of [10] on the necessity of interaction for economic efficiency.

An interesting direction for future research, also advocated by [10], is to consider the case where the bidders valuations are *submodular*. It is known that obtaining a better than $(1 - 1/2e)$-approximation to social welfare in submodular combinatorial auctions requires exponential communication [13] (regardless of the number of rounds of interaction). However, no better lower bounds are known for bounded-round protocols (even for simultaneous ones). Another interesting open problem is to close the gap between the $\Omega(\log\log n)$ lower bound of [2] and the $O(\log n)$ upper bound of [10] on the number of rounds necessary to achieve an efficient allocation in matching markets.

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