Localization with GPS

From GPS Theory and Practice Fifth Edition

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Introduction

- GPS = Global Positioning System
- Three segments:

1. Space (24 satellites)
2. Control (DOD)
3. User (civilian and military receivers)

GPS Overview

- Satellites transmit L1 and L2 signals
- L1--two pseudorandom noise signals
  - Protected (P-)code
  - Course acquisition (C/A) code (most civilian receivers)
- L2--P-code only
- Anti-spoofing adds noise to the P-code, resulting in Y-code

Observables

- Code pseudoranges

$$R = c\,\Delta t = c\,\Delta t(\mathrm{GPS}) + c\,\Delta\delta = \varrho + c\,\Delta\delta\,.$$

$$\begin{aligned}\varrho = \varrho(t^S, t_R) &= \varrho(t^S, (t^S + \Delta t)) \\ &= \varrho(t^S) + \dot{\varrho}(t^S)\,\Delta t\end{aligned}$$

Observables

- Phase pseudoranges
  - N = number of cycles between satellite and receiver

$$\Phi = \frac{1}{\lambda}\varrho + \frac{c}{\lambda}\Delta\delta + N$$

Observables

- Doppler Data
  - Dots indicate derivatives wrt time.

$$D = \lambda \dot{\Phi} = \dot{\varrho} + c\,\Delta\dot{\delta}$$

Observables

- Biases and Noise

Table 6.1. Range biases

Source	Effect
Satellite	Clock bias
	Orbital errors
Signal propagation	Ionospheric refraction
	Tropospheric refraction
Receiver	Antenna phase center variation
	Clock bias
	Multipath

Combining Observables

- Generally
- Linear combinations with integers
- Linear combinations with real numbers
- Smoothing

Mathematical Models for Positioning

- Point positioning
- Differential positioning
  - With code ranges
  - With phase ranges
- Relative positioning
  - Single differences
  - Double differences
  - Triple differences

Point Positioning

With Code Ranges

$$R_i^j(t) + c\,\delta^j(t) = \varrho_i^j(t) + c\,\delta_i(t)\,.$$

With Carrier Phases

$$\Phi_i^j(t) + f^j\,\delta^j(t) = \frac{1}{\lambda}\,\varrho_i^j(t) + N_i^j + f^j\,\delta_i(t)\,.$$

With Doppler Data

$$D_i^j(t) = \dot{\varrho}_i^j(t) + c\,\Delta\dot{\delta}_i^j(t)$$

Differential Positioning

Two receivers used:

- Fixed, A: Determines PRC and RRC
- Rover, B: Performs point pos'ning with PRC and RRC from A

With Code Ranges

$$R_A^j(t_0) = \varrho_A^j(t_0) + \Delta\varrho_A^j(t_0) + \Delta\varrho^j(t_0) + \Delta\varrho_A(t_0)$$

$$R_B^j(t)_{\text{corr}} = \varrho_B^j(t) + \Delta\varrho_{AB}(t)$$

Differential Positioning

With Phase Ranges

$$\lambda\,\Phi^j_A(t_0) = \varrho^j_A(t_0) + \Delta\varrho^j_A(t_0) + \Delta\varrho^j(t_0) + \Delta\varrho_A(t_0) + \lambda\,N^j_A$$

$$\lambda\,\Phi^j_B(t)_{\mathrm{corr}} = \varrho^j_B(t) + \Delta\varrho_{AB}(t) + \lambda\,N^j_{AB}$$

Relative Positioning

Aim is to determine the baseline vector A->B.

A is known,

B is the reference point

Assumptions: A, B are simultaneously observed

Single Differences:

- two points and one satellite
- Phase equation of each point is differenced to yield

$$\Phi^{j}_{AB}(t) = \frac{1}{\lambda}\varrho^{j}_{AB}(t) + N^{j}_{AB} + f^{j}\,\delta_{AB}(t)$$

Relative Positioning

- Double differences
  - Two points and two satellites
  - Difference of two single-differences gives

$$\Phi^{jk}_{AB}(t) = \frac{1}{\lambda}\varrho^{jk}_{AB}(t) + N^{jk}_{AB}\,.$$

Relative Positioning

- Triple-Differences
  - Difference of double-differences across two epochs

$$\Phi_{AB}^{jk}(t_{12}) = \frac{1}{\lambda}\,\varrho_{AB}^{jk}(t_{12})$$

Adjustment of Mathematical Models

- Models above need adjusting so that they are in a linear form.
- Idea is to linearize the distance metrics which carry the form:

$$\begin{aligned}\rho_i^j(t) &= \sqrt{(X^j(t) - X_i)^2 + (Y^j(t) - Y_i)^2 + (Z^j(t) - Z_i)^2} \\ &= f(X_i, Y_i, Z_i)\end{aligned}$$

Adjustment of Mathematical Models

- Each coordinate is decomposed as follows:

$$X_i = X_{i0} + \Delta X_i$$
$$Y_i = Y_{i0} + \Delta Y_i$$
$$Z_i = Z_{i0} + \Delta Z_i$$

Allowing the Taylor series expansion of f

$$f(X_i, Y_i, Z_i) \equiv f(X_{i0} + \Delta X_i,\ Y_{i0} + \Delta Y_i,\ Z_{i0} + \Delta Z_i)$$
$$= f(X_{i0}, Y_{i0}, Z_{i0}) + \frac{\partial f(X_{i0}, Y_{i0}, Z_{i0})}{\partial X_{i0}} \Delta X_i$$
$$+ \frac{\partial f(X_{i0}, Y_{i0}, Z_{i0})}{\partial Y_{i0}} \Delta Y_i + \frac{\partial f(X_{i0}, Y_{i0}, Z_{i0})}{\partial Z_{i0}} \Delta Z_i + \dots$$

Adjustment of Mathematical Models

- Computing the partial derivatives and substituting preliminary equations yields the linear equation:

$$\varrho_i^j(t) = \varrho_{i0}^j(t) - \frac{X^j(t) - X_{i0}}{\varrho_{i0}^j(t)}\Delta X_i - \frac{Y^j(t) - Y_{i0}}{\varrho_{i0}^j(t)}\Delta Y_i - \frac{Z^j(t) - Z_{i0}}{\varrho_{i0}^j(t)}\Delta Z_i$$

Linear Models

- Point Positioning with Code Ranges
  - Recall: $R_i^j(t) = \varrho_i^j(t) + c\,\delta_i(t) - c\,\delta^j(t)$
  - Substitution of the linearized term (prev. slide) and rearranging all unknowns to the left, gives:

$$R_i^j(t) - \varrho_{i0}^j(t) + c\,\delta^j(t) = -\frac{X^j(t) - X_{i0}}{\varrho_{i0}^j(t)}\,\Delta X_i - \frac{Y^j(t) - Y_{i0}}{\varrho_{i0}^j(t)}\,\Delta Y_i - \frac{Z^j(t) - Z_{i0}}{\varrho_{i0}^j(t)}\,\Delta Z_i + c\,\delta_i(t)$$

Linear Models

- Point Positioning with Code Ranges
- Four unknowns implies the need for four satellites. Let:

$$\ell^j = R_i^j(t) - \varrho_{i0}^j(t) + c\,\delta^j(t)$$

$$a_{X_i}^j = -\frac{X^j(t) - X_{i0}}{\varrho_{i0}^j(t)}$$

$$a_{Y_i}^j = -\frac{Y^j(t) - Y_{i0}}{\varrho_{i0}^j(t)}$$

$$a_{Z_i}^j = -\frac{Z^j(t) - Z_{i0}}{\varrho_{i0}^j(t)}$$

Linear Models

- Point Positioning with Code Ranges
- Assuming satellites numbered from 1 to 4

$$\ell^1 = a^1_{X_i}\,\Delta X_i + a^1_{Y_i}\,\Delta Y_i + a^1_{Z_i}\,\Delta Z_i + c\,\delta_i(t)$$

$$\ell^2 = a^2_{X_i}\,\Delta X_i + a^2_{Y_i}\,\Delta Y_i + a^2_{Z_i}\,\Delta Z_i + c\,\delta_i(t)$$

$$\ell^3 = a^3_{X_i}\,\Delta X_i + a^3_{Y_i}\,\Delta Y_i + a^3_{Z_i}\,\Delta Z_i + c\,\delta_i(t)$$

$$\ell^4 = a^4_{X_i}\,\Delta X_i + a^4_{Y_i}\,\Delta Y_i + a^4_{Z_i}\,\Delta Z_i + c\,\delta_i(t)$$

Superscripts denote satellite numbers, not indices.

Linear Models

Point Positioning with Code Ranges

- We can now express the model in matrix form as $l = Ax$ where

$$\underline{A} = \begin{bmatrix} a^1_{X_i} & a^1_{Y_i} & a^1_{Z_i} & c \\ a^2_{X_i} & a^2_{Y_i} & a^2_{Z_i} & c \\ a^3_{X_i} & a^3_{Y_i} & a^3_{Z_i} & c \\ a^4_{X_i} & a^4_{Y_i} & a^4_{Z_i} & c \end{bmatrix} \qquad \underline{x} = \begin{bmatrix} \Delta X_i \\ \Delta Y_i \\ \Delta Z_i \\ \delta_i(t) \end{bmatrix} \qquad \underline{\ell} = \begin{bmatrix} \ell^1 \\ \ell^2 \\ \ell^3 \\ \ell^4 \end{bmatrix}$$

Linear Models

Point Positioning with Carrier Phases

- Similarly computed.
- Ambiguities in the model raise the number of unknowns from 4 to 8
- Need three epochs to solve the system. It produces 12 equations with 10 unknowns.

Linear Models

Point Positioning with Carrier Phases

Linear Model

$$\lambda\,\Phi_i^j(t) - \varrho_{i0}^j(t) + c\,\delta^j(t) = -\frac{X^j(t) - X_{i0}}{\varrho_{i0}^j(t)}\,\Delta X_i - \frac{Y^j(t) - Y_{i0}}{\varrho_{i0}^j(t)}\,\Delta Y_i - \frac{Z^j(t) - Z_{i0}}{\varrho_{i0}^j(t)}\,\Delta Z_i + \lambda\,N_i^j + c\,\delta_i(t)$$

Linear Models

Point Positioning with Carrier Phases

$l = Ax$

$$\underline{\ell} = \begin{bmatrix} \lambda\,\Phi_i^1(t) - \varrho_{i0}^1(t) + c\,\delta^1(t) \\ \lambda\,\Phi_i^2(t) - \varrho_{i0}^2(t) + c\,\delta^2(t) \\ \lambda\,\Phi_i^3(t) - \varrho_{i0}^3(t) + c\,\delta^3(t) \\ \lambda\,\Phi_i^4(t) - \varrho_{i0}^4(t) + c\,\delta^4(t) \end{bmatrix}$$

$$\underline{A} = \begin{bmatrix} a_{X_i}^1(t) & a_{Y_i}^1(t) & a_{Z_i}^1(t) & \lambda & 0 & 0 & 0 & c \\ a_{X_i}^2(t) & a_{Y_i}^2(t) & a_{Z_i}^2(t) & 0 & \lambda & 0 & 0 & c \\ a_{X_i}^3(t) & a_{Y_i}^3(t) & a_{Z_i}^3(t) & 0 & 0 & \lambda & 0 & c \\ a_{X_i}^4(t) & a_{Y_i}^4(t) & a_{Z_i}^4(t) & 0 & 0 & 0 & \lambda & c \end{bmatrix}$$

$$\underline{x}^T = \begin{bmatrix} \Delta X_i & \Delta Y_i & \Delta Z_i & N_i^1 & N_i^2 & N_i^3 & N_i^4 & \delta_i(t) \end{bmatrix}$$

Linear Models

Relative Positioning

- Carrier phases considered
- Double-differences treated
- Recall: DD equation * _

$$\lambda \Phi_{AB}^{jk}(t) = \varrho_{AB}^{jk}(t) + \lambda N_{AB}^{jk}$$

- The second term on the lhs is expanded and linearized as in previous models to yield:

Linear Models

Relative Positioning

- The second term on the lhs is expanded and linearized as in previous models to yield ([9.133]…see paper pg 262)
- *l*'s:

$$\begin{aligned} \ell_{AB}^{jk}(t) = \; & a_{X_A}^{jk}(t)\,\Delta X_A + a_{Y_A}^{jk}(t)\,\Delta Y_A + a_{Z_A}^{jk}(t)\,\Delta Z_A \\ & + a_{X_B}^{jk}(t)\,\Delta X_B + a_{Y_B}^{jk}(t)\,\Delta Y_B + a_{Z_B}^{jk}(t)\,\Delta Z_B + \lambda\,N_{AB}^{jk} \end{aligned}$$

Linear Models

Relative Positioning

- The right hand side is abbreviated as follows (a’s):

$$a_{Y_A}^{jk}(t) = +\frac{Y^k(t) - Y_{A0}}{\varrho_{A0}^k(t)} - \frac{Y^j(t) - Y_{A0}}{\varrho_{A0}^j(t)}$$

$$a_{Z_A}^{jk}(t) = +\frac{Z^k(t) - Z_{A0}}{\varrho_{A0}^k(t)} - \frac{Z^j(t) - Z_{A0}}{\varrho_{A0}^j(t)}$$

$$a_{X_B}^{jk}(t) = -\frac{X^k(t) - X_{B0}}{\varrho_{B0}^k(t)} + \frac{X^j(t) - X_{B0}}{\varrho_{B0}^j(t)}$$

$$a_{Y_B}^{jk}(t) = -\frac{Y^k(t) - Y_{B0}}{\varrho_{B0}^k(t)} + \frac{Y^j(t) - Y_{B0}}{\varrho_{B0}^j(t)}$$

$$a_{Z_B}^{jk}(t) = -\frac{Z^k(t) - Z_{B0}}{\varrho_{B0}^k(t)} + \frac{Z^j(t) - Z_{B0}}{\varrho_{B0}^j(t)}$$

Linear Models

Relative Positioning

- Since the coordinates of A must be known, the number of unknowns is reduced by three. Now, 4 satellites (*j,k,l,m)* and two epochs are needed to solve the system.

Extra References

- Introduction and overview: http://www.gpsy.org/gpsinfo/gps-faq.txt