Hidden Markov Models II

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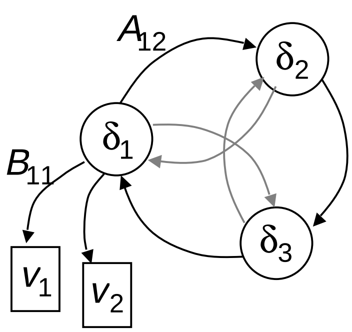
CS 520, Spring 2000

Lecture 9

HMM – Recap

- **Models based on key independence assumptions for time-series data**



Events:

$$\delta_i^{(t)} \qquad v_k^{(t)}$$

Arcs determine matrix A

$$A_{ij} = P(\delta_j^{(t)} \mid \delta_i^{(t-1)})$$

Obs governed by matrix B

$$B_{jk} = P(v_k^{(t)} \mid \delta_j^{(t)})$$

HMM – Recap

- **Basic event: seeing (given) observations when system follows (hypothesized) path**

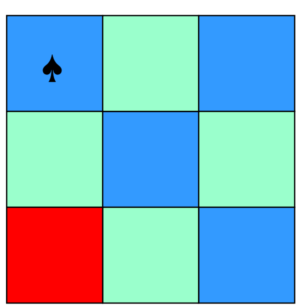
$$P(\mathbf{v},\delta) = L^{[\delta]} \prod_{u=2}^{m} A^{[\delta,u]} \prod_{u=1}^{m} B^{[\mathbf{v},\delta,u]}$$

- **We started with the evaluation problem**

Compute $P(\mathbf{v} \mid \omega_i, \text{len} = m)$

Example

- **Track robot motion in a 3x3 grid of rooms:**



- **Robot moves randomly to adjacent rooms**
- **Rooms have either red, green or blue walls**
  - color is observed
  - start at ♠

Example
(CONTINUED)

- **Observe sequence:**
  - blue (b), green (g), red (r)
- **Question: are you in this environment** (ω_0)
  - (or some other?)
- **Answer using evaluation:**

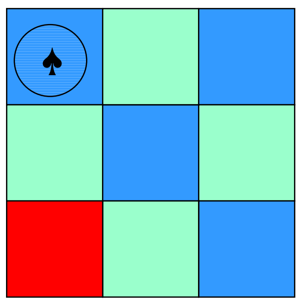
$$P(b^{(1)}g^{(2)}r^{(3)} \mid \omega_0)$$

Step Through Evaluation

- **Build a table**
  - of the likelihood of being in room *r* at time *t* given the observations so far

Step Through Evaluation 1

- **Start with the first step**



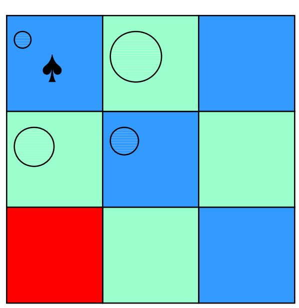
$$P(r_i^{(1)}, \mathrm{b}^{(1)}) = L_i B_{i\mathrm{b}} = \begin{cases} 0.9 & \text{for start state} \\ 0 & \text{otherwise} \end{cases}$$

Assumes color confusion:

	b	g	r
b	0.87	0.1	0.03
g	0.1	0.87	0.03
r	0.03	0.03	0.94

Step Through Evaluation 2

- **Sum up transitions to nearby states**



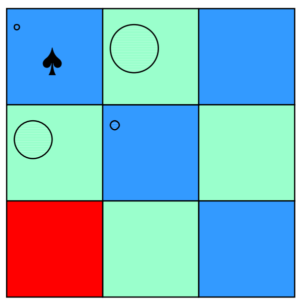
$$P(r_j^{(2)}, \mathrm{b}^{(1)}) = \sum_i A_{ij} P(r_i^{(1)}, \mathrm{b}^{(1)})$$

Assumes transition matrix:

move	00	01	10	11
p	0.05	0.3	0.5	0.15

Step Through Evaluation 3

- **Factor in second observation**



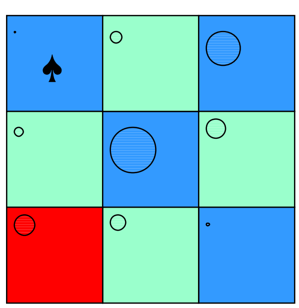
$$P(r_j^{(2)}, \mathrm{b}^{(1)}, \mathrm{g}^{(2)}) = B_{j\mathrm{g}} P(r_j^{(2)}, \mathrm{b}^{(1)})$$

Apply confusion matrix:

	b	g	r
b	0.87	0.1	0.03
g	0.1	0.87	0.03
r	0.03	0.03	0.94

Step Through Evaluation 4

- **Sum up transitions to nearby states**



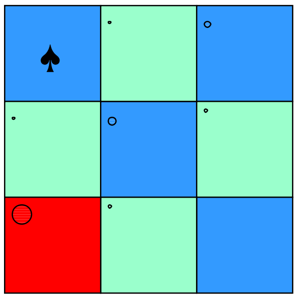
$$P(r_k^{(3)}, \mathrm{b}^{(1)}, \mathrm{g}^{(2)}) = \sum_j A_{jk} P(r_j^{(2)}, \mathrm{b}^{(1)}, \mathrm{g}^{(2)})$$

Assumes transition matrix:

move	00	01	10	11
p	0.05	0.3	0.5	0.15

Step Through Evaluation 5

- **Factor in third observation**



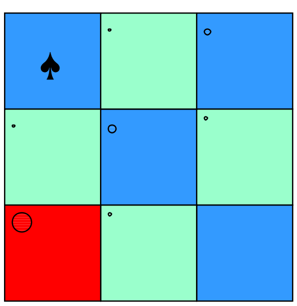
$$P(r_k^{(3)}, b^{(1)}, g^{(2)}, r^{(3)}) = B_{kr} P(r_k^{(3)}, b^{(1)}, g^{(2)})$$

Apply confusion matrix:

	b	g	r
b	0.87	0.1	0.03
g	0.1	0.87	0.03
r	0.03	0.03	0.94

Step Through Evaluation 6

- **Sum up to account for observations**



$$P(b^{(1)}, g^{(2)}, r^{(3)}) = \sum_k P(r_k^{(3)}, b^{(1)}, g^{(2)}, r^{(3)})$$

$$= \bullet$$

Forward Algorithm

- **Key points:**
  - For each new step, only the state at the last step (and the probability we ended there) is needed
  - Sum over all state sequences using dynamic programming
  - Finish by summing out over possible final states

HMM – Recap

- **We have seen the evaluation problem**

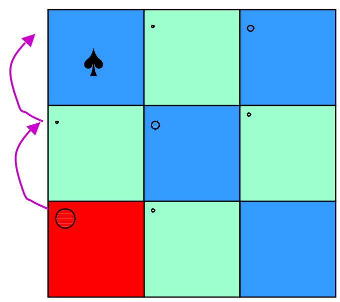
  Compute $P(\mathbf{v} \mid \omega_i, \text{len} = m)$

- **Now we turn to the decoding problem**

  Find $\underset{\delta}{\text{argmax}}\ P(\delta \mid \mathbf{v}, \omega_i)$

Decoding Example

- **For our robot from before –**



$$\arg\max_{\delta} P(\delta \mid b^{(1)}, g^{(2)}, r^{(3)})$$

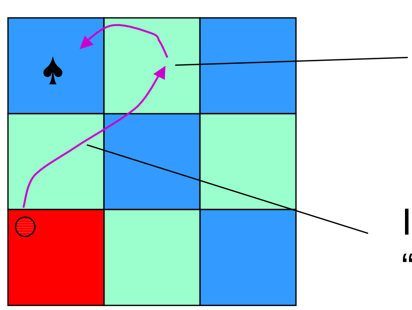
$$= \arg\max_{\delta} \frac{P(\delta, b^{(1)}, g^{(2)}, r^{(3)})}{P(b^{(1)}, g^{(2)}, r^{(3)})}$$

$$= \arg\max_{\delta} P(\delta, b^{(1)}, g^{(2)}, r^{(3)})$$

$$= \bigcirc$$

Decoding vs. Evaluation

- **The sequence of most likely states from evaluation is different:**

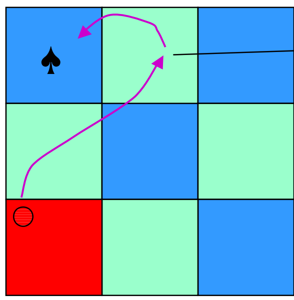


Most likely state at time 2

Impossible transition, so "very unlikely" sequence

Decoding vs. Evaluation

- **Part of the difference is that the forward algorithm only uses past observations**



- Most likely state at time 2, given 2 observations
- Not likely state for time 2, given 3 observations

Decoding vs. Evaluation

- **Part of the difference is that the forward algorithm only uses past observations**
- **We could get around this**
  - by assuming system is in state *i* at time *t*, and reapplying forward algorithm onward
  - by computing full distribution on states at time *t* given future info, via the backward algorithm (as we'll see Wednesday)

Decoding vs. Evaluation

- **Part of the difference is that the forward algorithm only uses past observations**
- **We could get around this**

- **But decoding is different from evaluation in another way…**

Decoding vs. Evaluation

$$\text{Let } \hat{\delta} = \underset{\delta}{\operatorname{argmax}}\ P(\delta \mid \mathbf{v})$$

- **Overall likelihood of being in state given observations:**

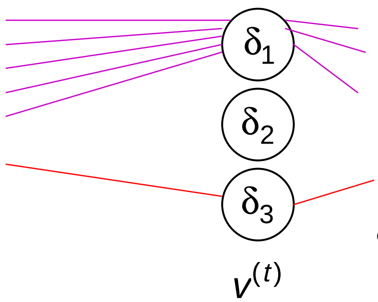
$$P(\delta_j^{(t)}, \mathbf{v})$$

- **Do not have:**

$$\hat{\delta}^{(t)} = \underset{\delta_j}{\operatorname{argmax}}\, P(\delta_j^{(t)}, \mathbf{v})$$

Decoding vs. Evaluation
VISUALIZATION

- **In words:**
  – State t of the highest probability sequence need not have highest probability at t



Many low probability paths can give high

$P(\delta_1^{(t)}, \mathbf{v})$

and outweigh single high probability path at δ_3

Viterbi Decoding

- **Tabling also works for decoding**
- **As with evaluation**
  – Unfold the HMM through time
  – Assign a value to each state at each step
- **For decoding, value is**
  – Most likely state sequence up to there
  – Probability of that sequence and past observations

Formal Justification

- **Key probabilities to maximize:**

  $P(\delta^{(\le t)}, \mathbf{v}^{(\le t)})$ subject to $\delta^{(t)} = \delta_j$

  $$= A^{[\delta,t]} B^{[\mathbf{v},\delta,t]} P(\delta^{(\le t-1)}, \mathbf{v}^{(\le t-1)})$$

  maximized at $M_{j,t}$

- **Any pair** $\delta^{(t-1)}, \delta^{(t)}$ **determines** $A^{[\delta,t]} B^{[\mathbf{v},\delta,t]}$

Formal Justification
(CONTINUED)

- **So we conclude**

  $\hat{\delta}$ maximizes $P(\delta^{(\le t)}, \mathbf{v}^{(\le t)})$ subject to $\hat{\delta}^{(t)} = \delta_j$

- **Exactly when**

  $\hat{\delta}$ maximizes $P(\delta^{(\le t-1)}, \mathbf{v}^{(\le t-1)})$ subject to $\hat{\delta}^{(t-1)} = \delta_i$

  $\left(\text{at } P(\delta^{(\le t-1)}, \mathbf{v}^{(\le t-1)}) = M_{i,t-1}\right)$

- **And**

  $A_{ij} B_j^{[\mathbf{v},t]} M_{i,t-1}$ exceeds other $A_{i'j} B_j^{[\mathbf{v},t]} M_{i',t-1}$

Viterbi Algorithm

- **Initialize**
  - Set $M_{i,1} = L_i B_i^{[\mathbf{v},1]}$
  - Set best seq$(i,\ 1) = \delta_i^{(1)}$
- **Step**
  - Set $h = \arg\max_i A_{ij} B_j^{[\mathbf{v},t]} M_{i,t-1}$
  - Set $M_{j,t} = A_{hj} B_j^{[\mathbf{v},t]} M_{h,t-1}$
  - Set best seq$(j,\ t) = $ best seq$(h, t-1), \delta_j^{(t)}$

Viterbi Algorithm
(CONTINUED)

- **Finish**
  - After step m,
  - Set
    $$h = \arg\max_i M_{i,m}$$
  - Return best seq$(h,\ m)$

Classic Illustration of Viterbi

- **Part-of-speech tagging**
  - Preprocessing step in natural language processing
- **Words are ambiguous**
  - They may fulfill different roles in a sentence
  - Each role may be used with different senses of the word

Lexical Ambiguity

- **Here's an example of the contrast:**
  - Same word can provide object or action
    - The plants grow in Sandy's yard.
    - Sandy plants tomatoes in the yard.
  - But may describe different objects too
    - The plants bear fruit in August.
    - The plants employ union workers.

Part-of-speech Tagging

- **Research shows**
  - you can identify the right word sense well – if you know the role it plays in sentence – its part of speech
  - you can do a good job predicting parts of speech using local (Markov) models of word sequences

POS Tagging

- **Simplest case: bigram tagging**
  - (hidden) states are just parts of speech
  - observations are words
  - HMM parameter **A** gives probabilities of seeing parts of speech in succession
  - HMM parameter **B** gives probabilities of seeing words with different parts of speech
- **Trigram tagging typically used in practice**

POS Tagging

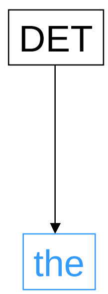
- **Use decoding**
  - Given a string of words **v**
  - Find sequence of parts of speech to maximize $P(\mathbf{v},\delta)$

Illustration

- **Process the example:**
  - Q: Why is this ski run difficult?
  - A: The slopes fall fast.

Processing Sketch

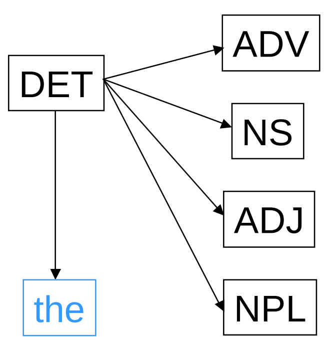
- **Initialize**



Many initial states are possible
But *the* can only be used as a determiner
So only one nonzero entry for $M_{i,1}$

Processing Sketch
(CONTINUED)

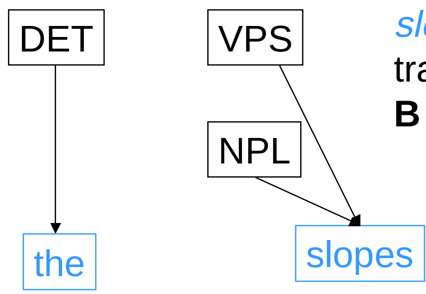
- **Next Step**



Determiners may be followed by a range of parts of speech
This translates to a number of nonzero **A** entries

Processing Sketch
(CONTINUED)

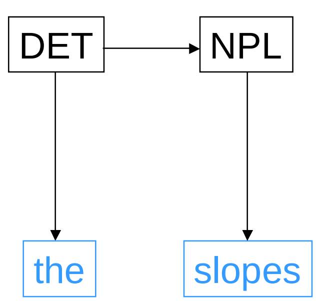
- **Step 2, continued**



And there are a range of ways to use *slopes*, which translate to nonzero **B** entries

Processing Sketch
(CONTINUED)

- **Step 2, continued**

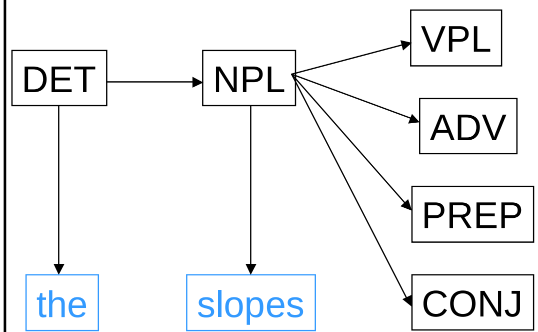


When we multiply **A** and **B** entries, the choice is clear

We save sequence DET, NPL and its probability

Processing Sketch
(CONTINUED)

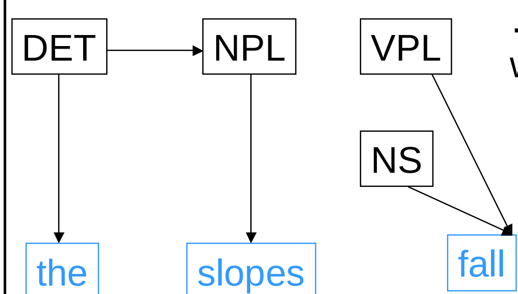
- **Next Step**



We consider what follows plural nouns…

Processing Sketch
(CONTINUED)

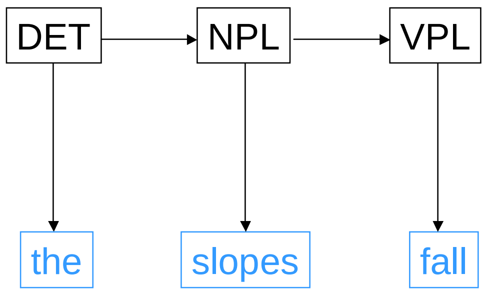
- **Step 3, continued**



…and the range of ways to use *fall*…

Processing Sketch
(CONTINUED)

- **Step 3, continued**



…to understand the next word.

We save sequence DET, NPL, VPL and its probability