Matching and Resource Allocation

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Rensselaer

Nobel prize in Economics 2013

Alvin E. Roth

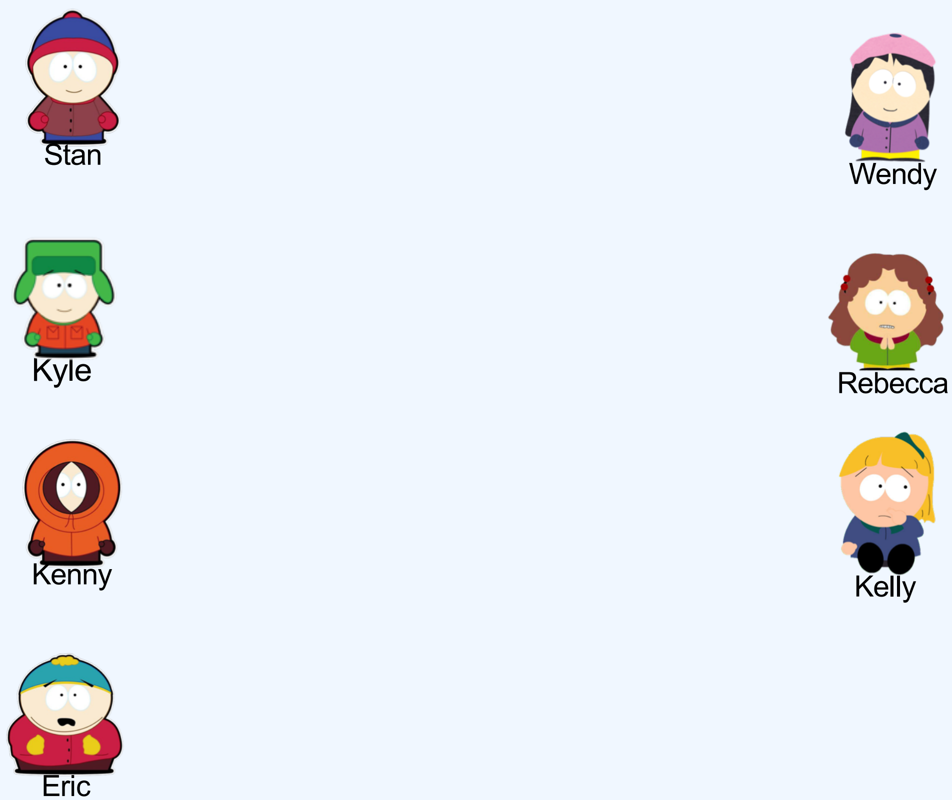
Lloyd Shapley

- "for the theory of stable allocations and the practice of market design."

Two-sided one-one matching

Boys

Girls

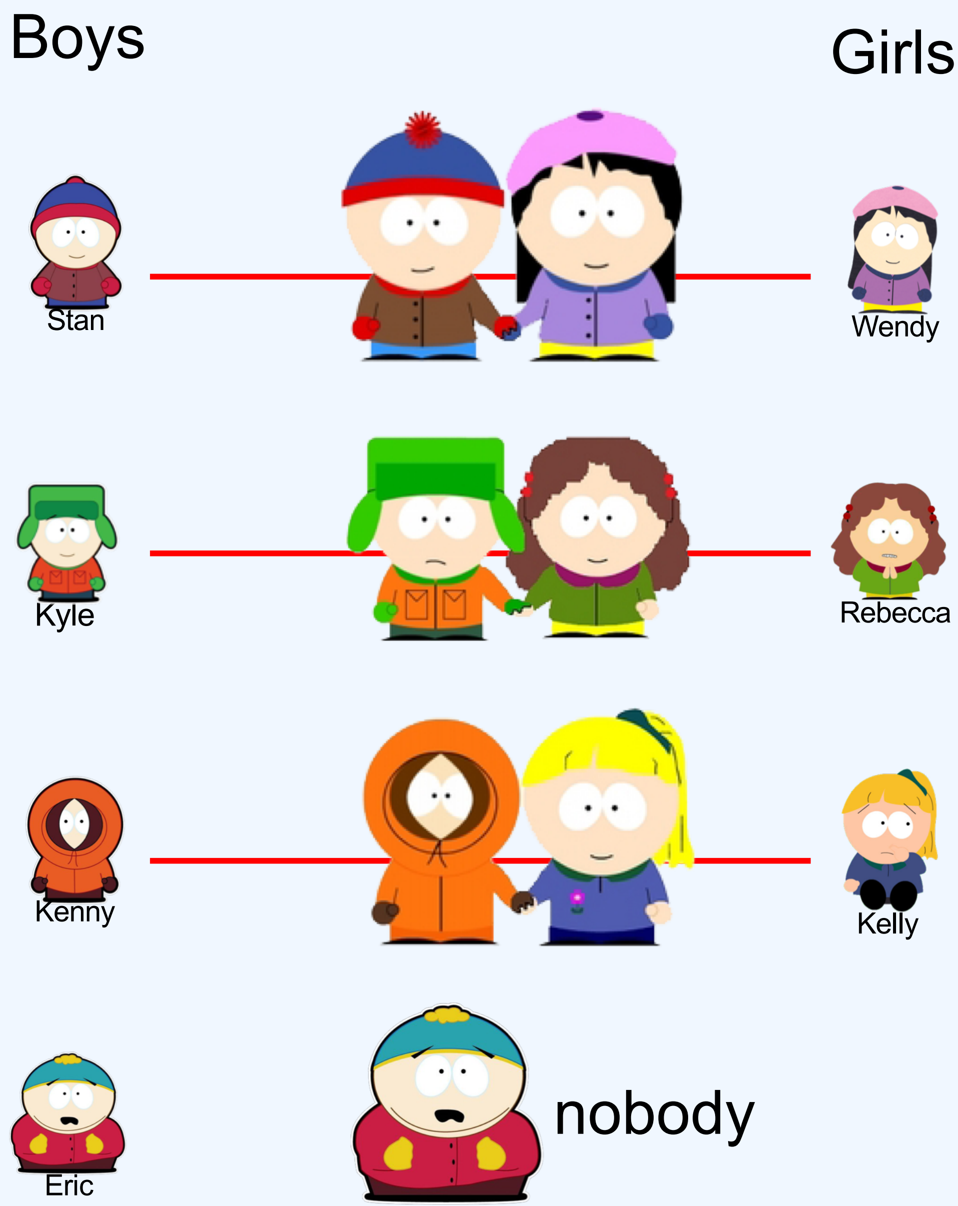


Applications: student/hospital, National Resident Matching Program

Formal setting

- Two groups: B and G
- Preferences:
  - members in B: full ranking over $G\cup$\{nobody\}
  - members in G: full ranking over $B\cup$\{nobody\}
- Outcomes: a matching M: $B\cup G\rightarrow B\cup G\cup$\{nobody\}
  - $M(B) \subseteq G\cup$\{nobody\}
  - $M(G) \subseteq B\cup$\{nobody\}
  - $[M(a)=M(b)\neq \text{nobody}] \Rightarrow [a=b]$
  - $[M(a)=b] \Rightarrow [M(b)=a]$

Example of a matching



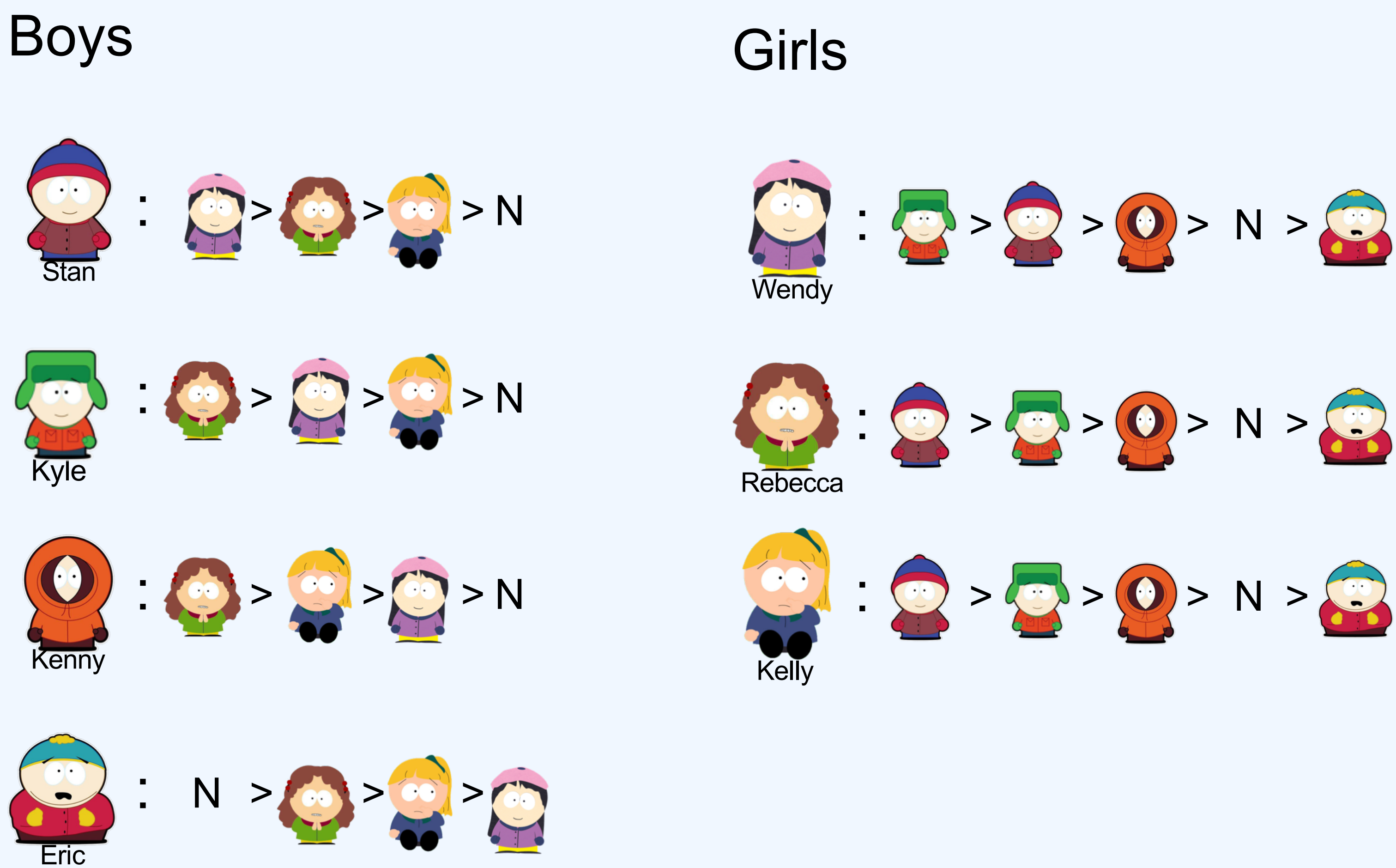
Good matching?

- Does a matching always exist?
  - apparently yes
- Which matching is the best?
  - utilitarian: maximizes “total satisfaction”
  - egalitarian: maximizes minimum satisfaction
  - but how to define utility?

Stable matchings

- Given a matching M, (b,g) is a blocking pair if
  - $g >_b M(b)$
  - $b >_g M(g)$
  - ignore the condition for nobody
- A matching is stable, if there is no blocking pair
  - no (boy,girl) pair wants to deviate from their currently matches

Example

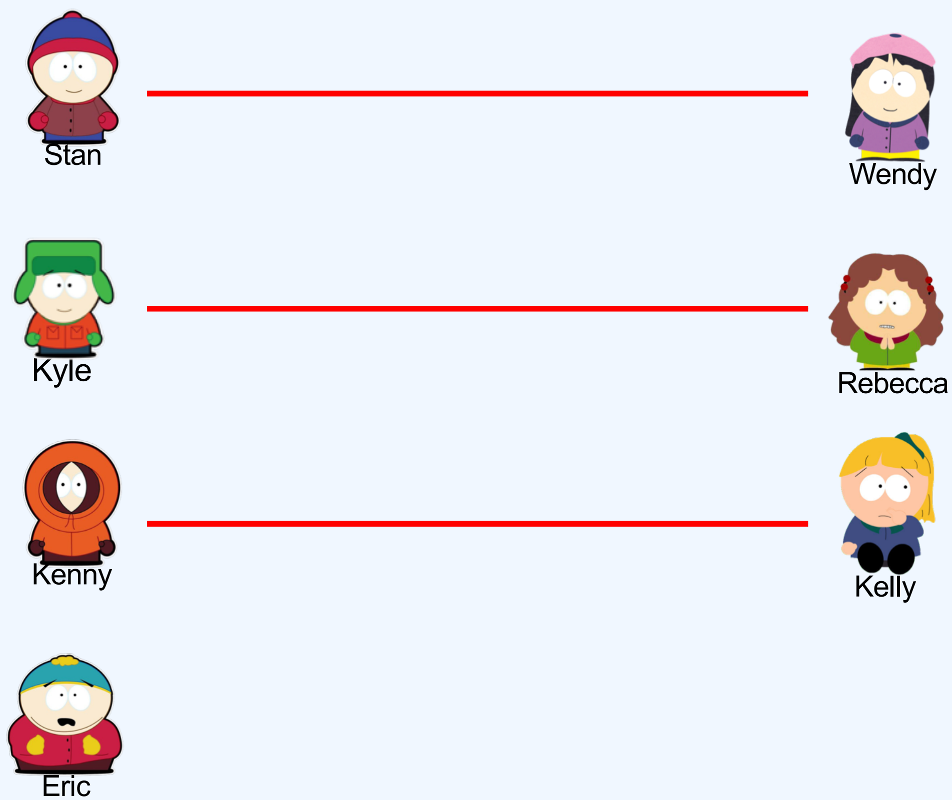


Boys
Girls
Stan
: > > > N
Kyle
: > > > N
Kenny
: > > > N
Eric
: N > > >
Wendy
: > > > N >
Rebecca
: > > > N >
Kelly
: > > > N >

A stable matching

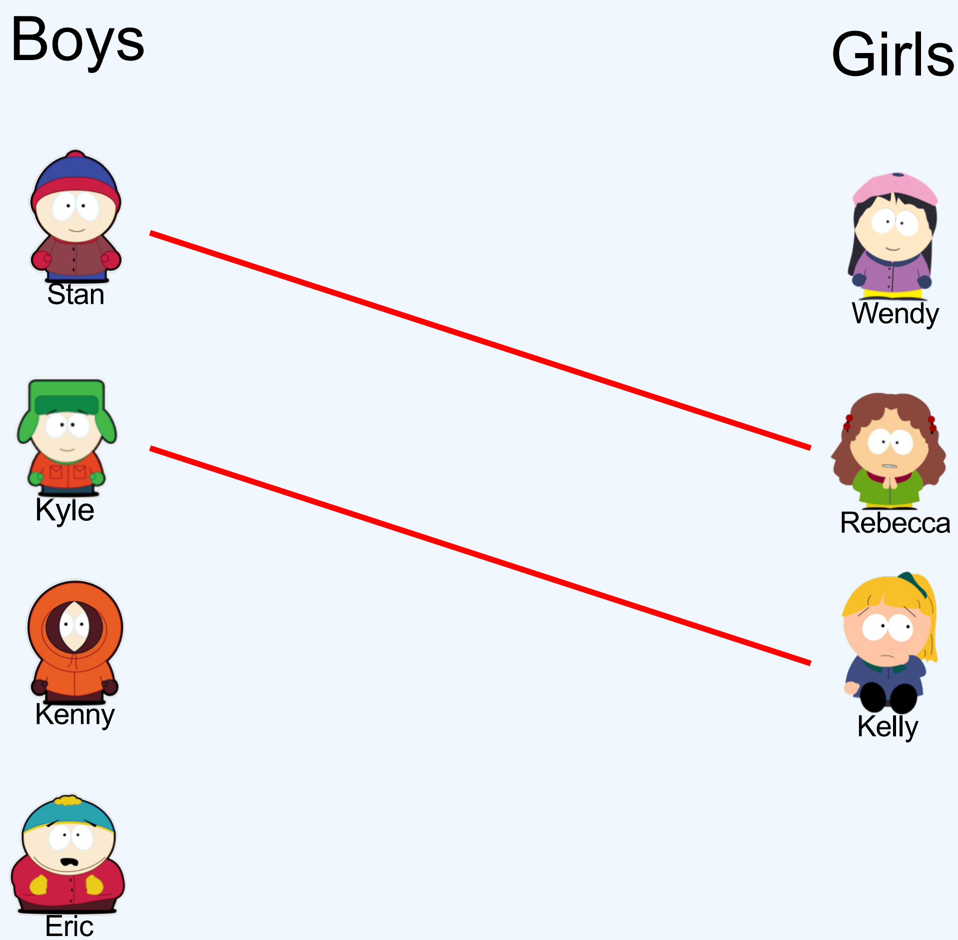
Boys

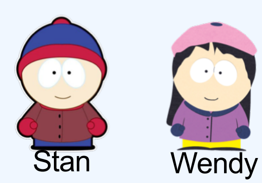
Girls



no link = matched to “nobody”

An unstable matching



Blocking pair: ()

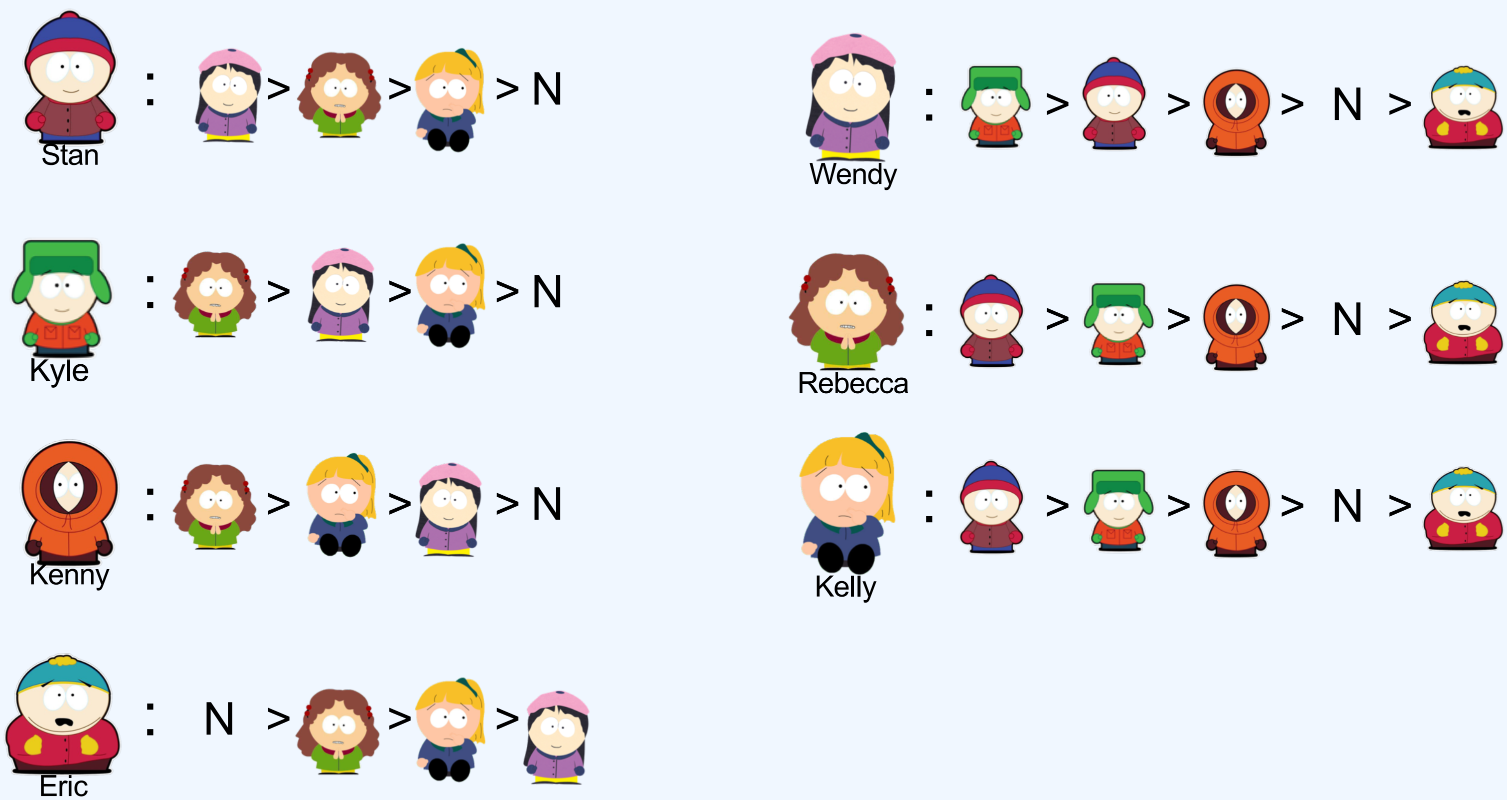
Does a stable matching always exist?

- Yes: Gale-Shapley's deferred acceptance algorithm (DA)
- Men-proposing DA: each girl starts with being matched to "nobody"
  - each boy proposes to his top-ranked girl (or "nobody") who has not rejected him before
  - each girl rejects all but her most-preferred proposal
  - until no boy can make more proposals
- In the algorithm
  - Boys are getting worse
  - Girls are getting better

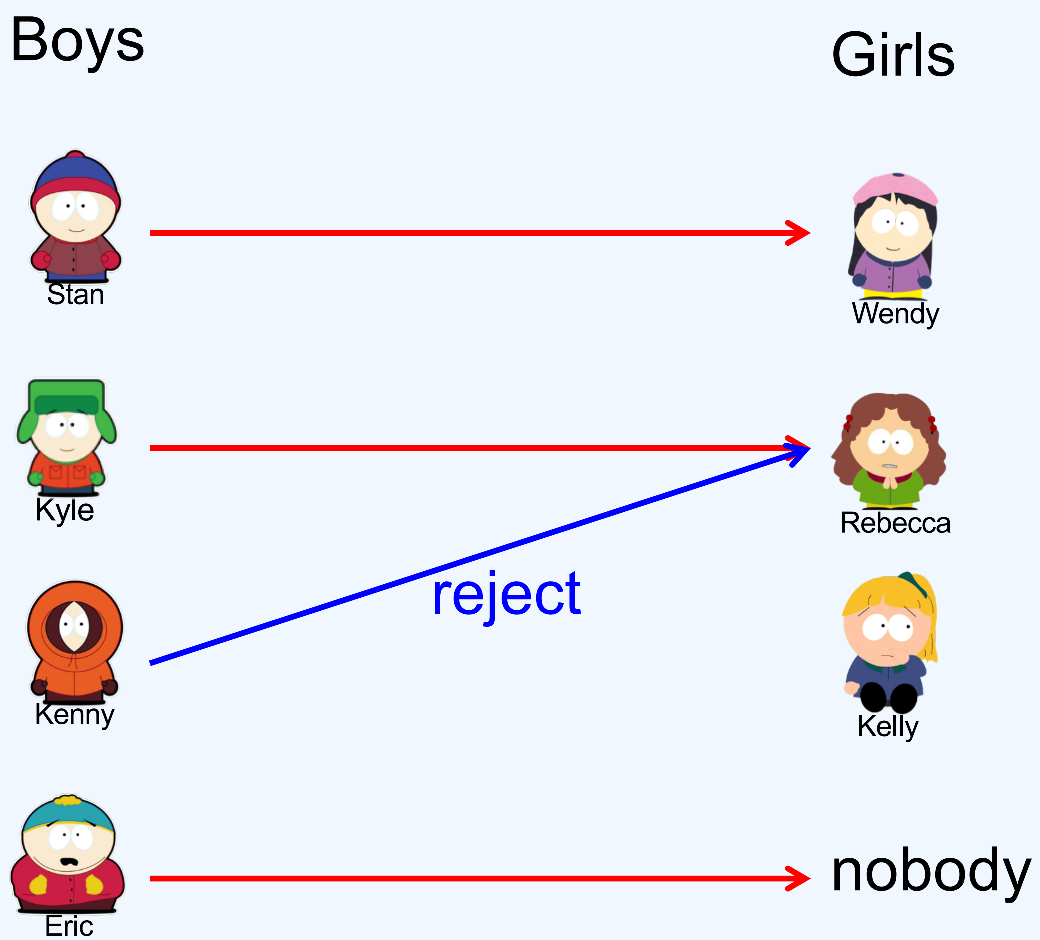
Men-proposing DA (on blackboard)

Boys

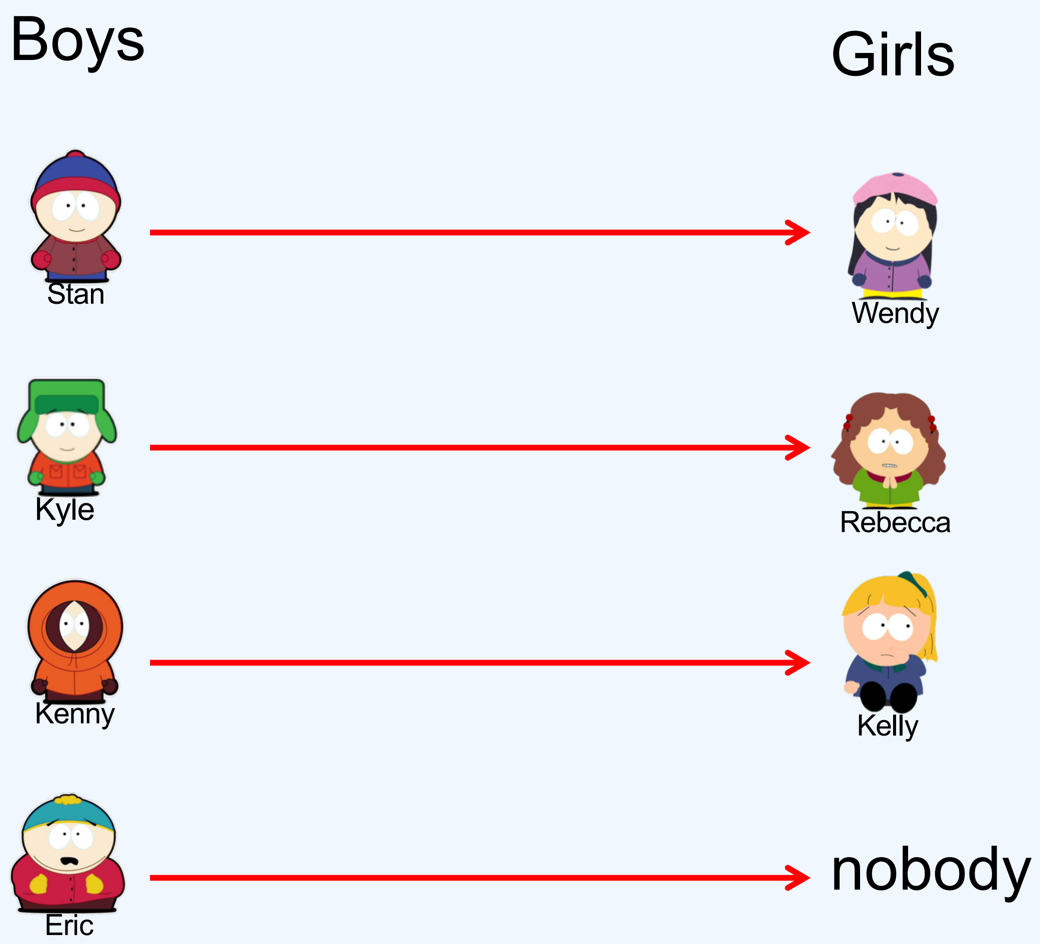
Girls



Round 1


Boys
Girls
Stan
Wendy
Kyle
Rebecca
reject
Kenny
Kelly
Eric
nobody

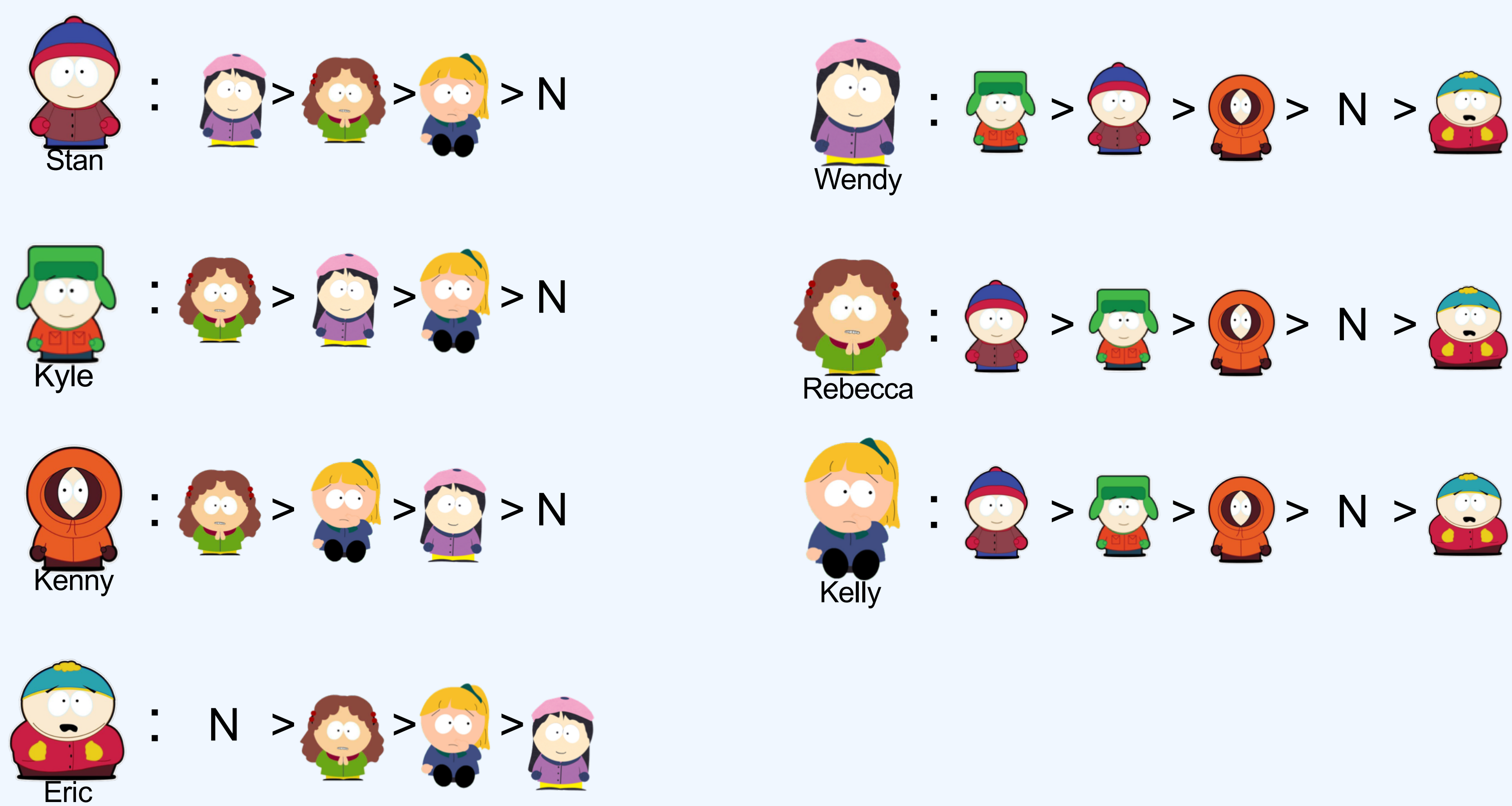
Round 2



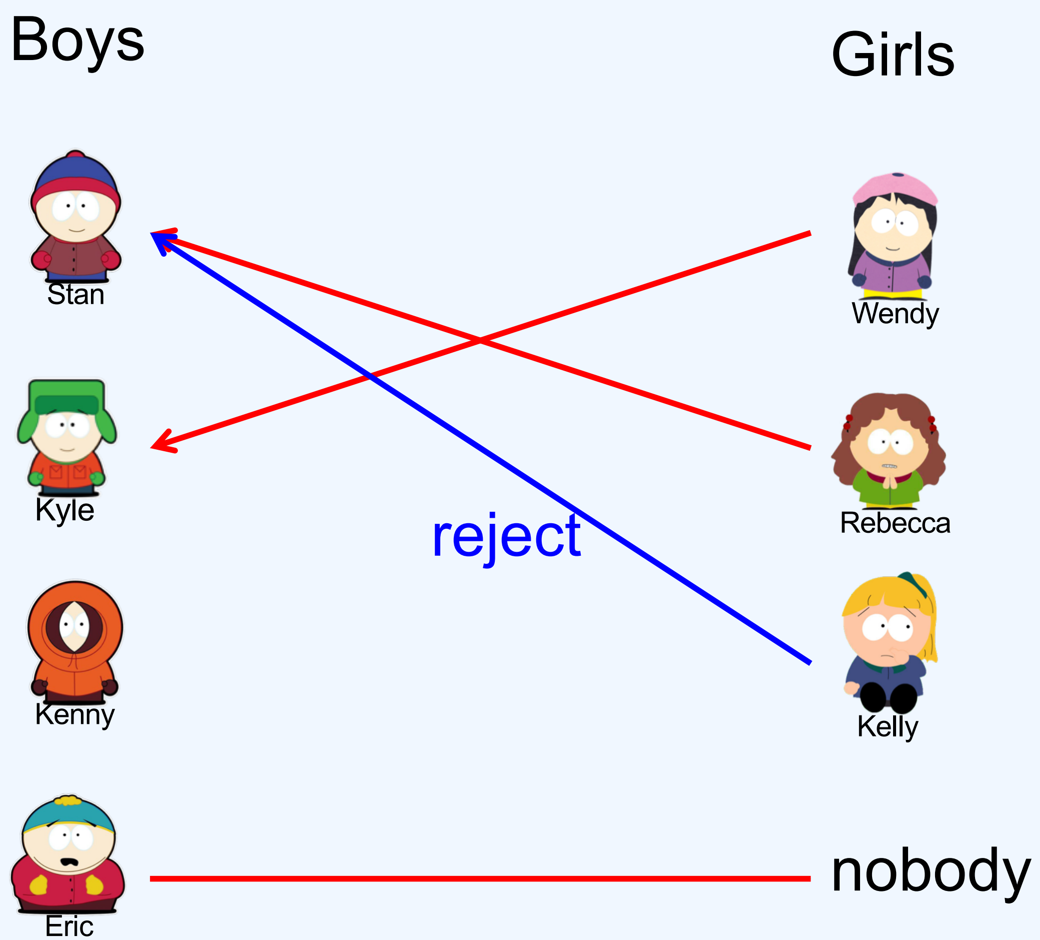
Women-proposing DA (on blackboard)

Boys

Girls



Round 1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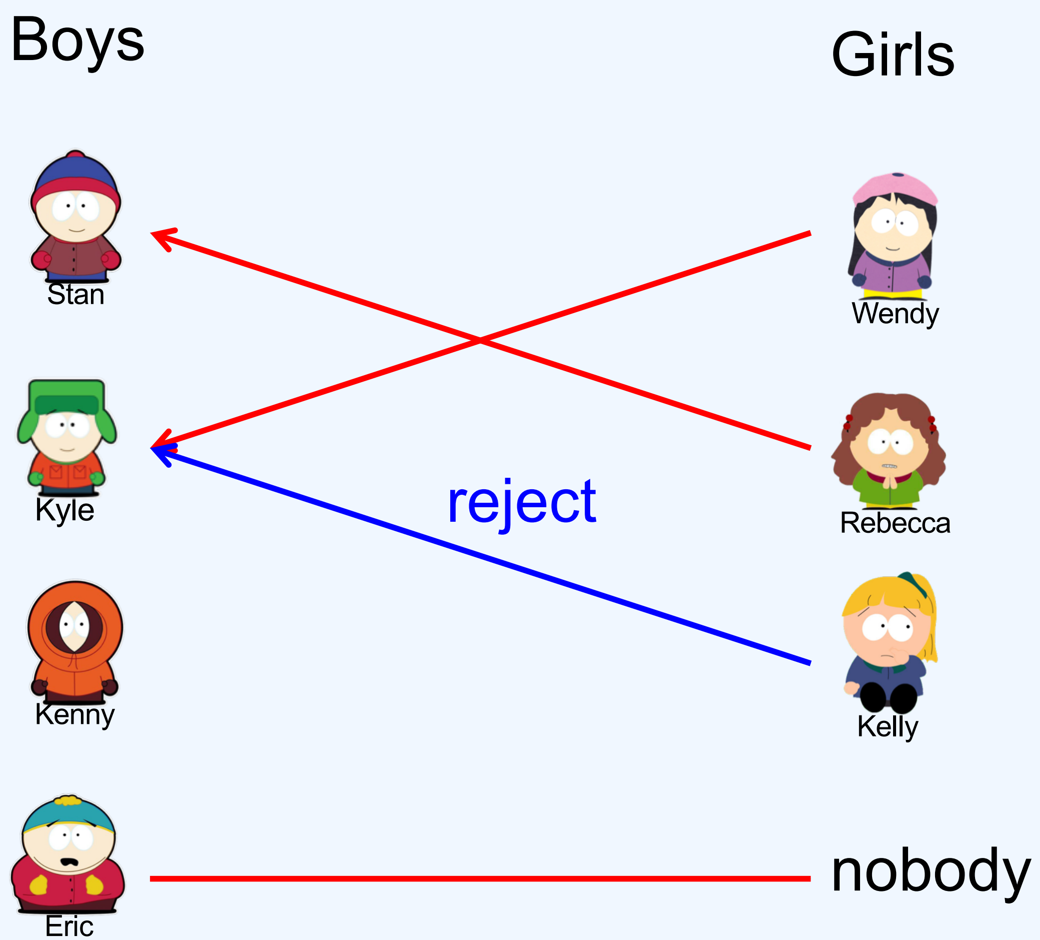
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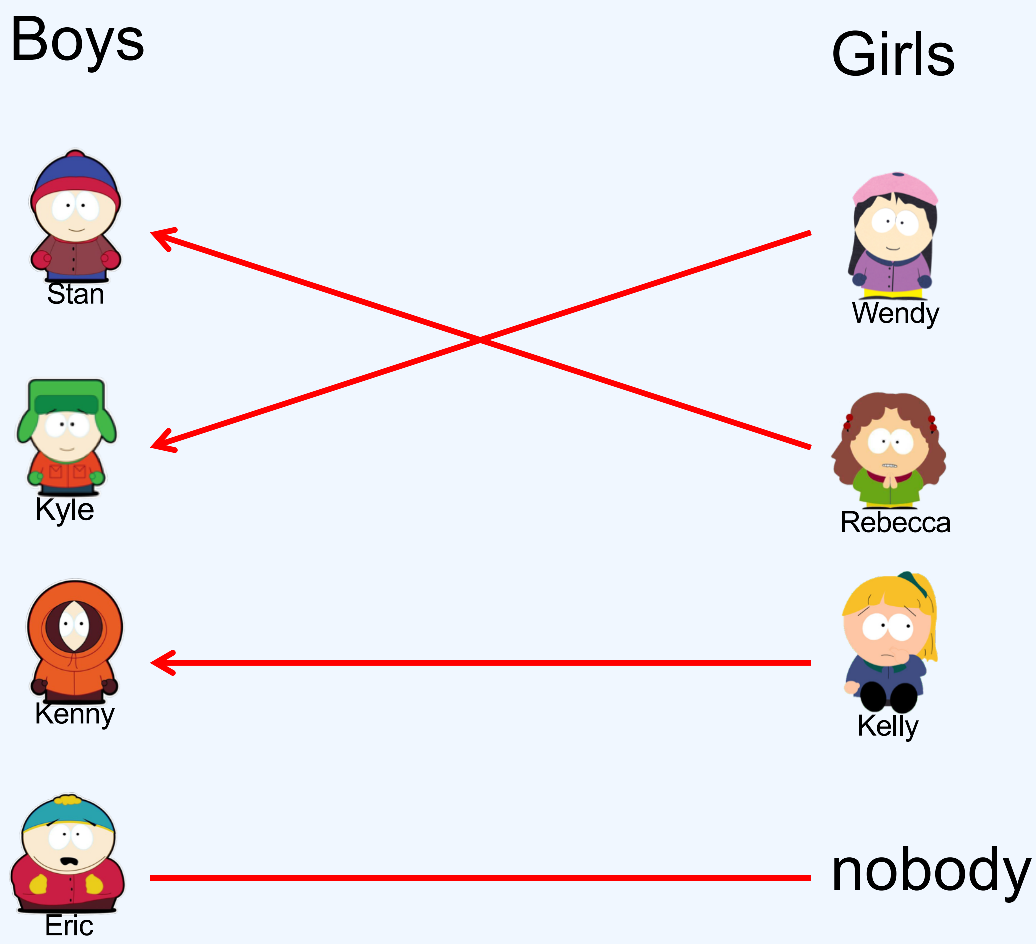
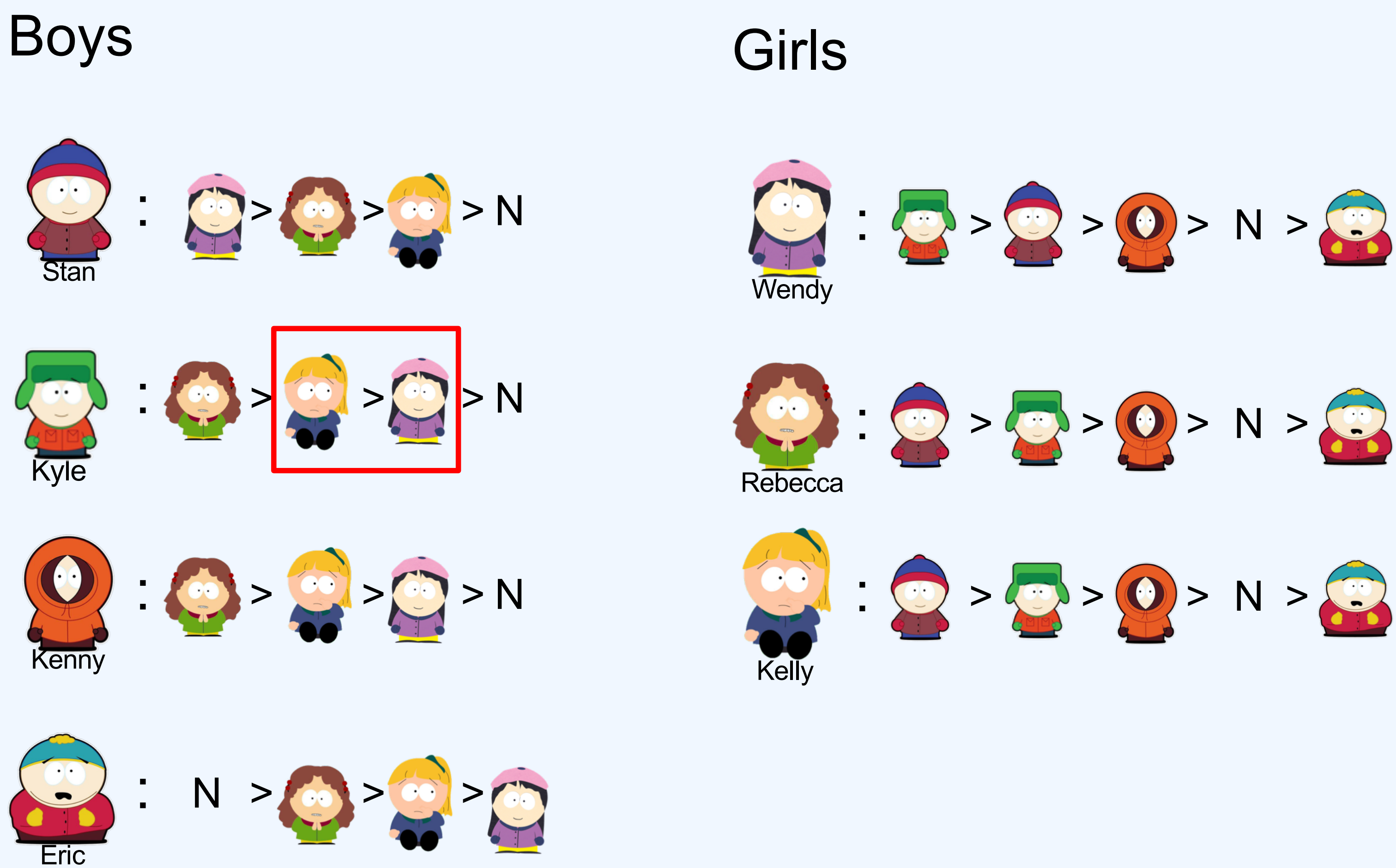
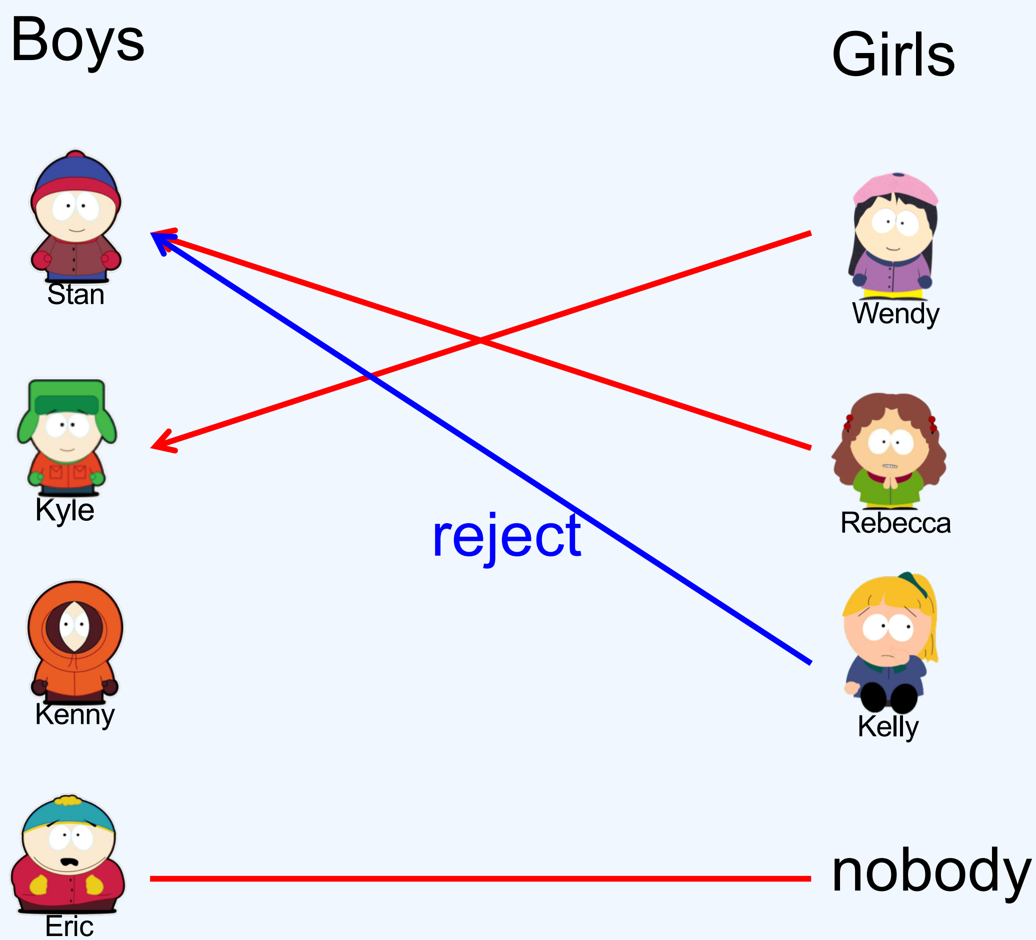
Round 3



Women-proposing DA with slightly different preferences



Round 1



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reject
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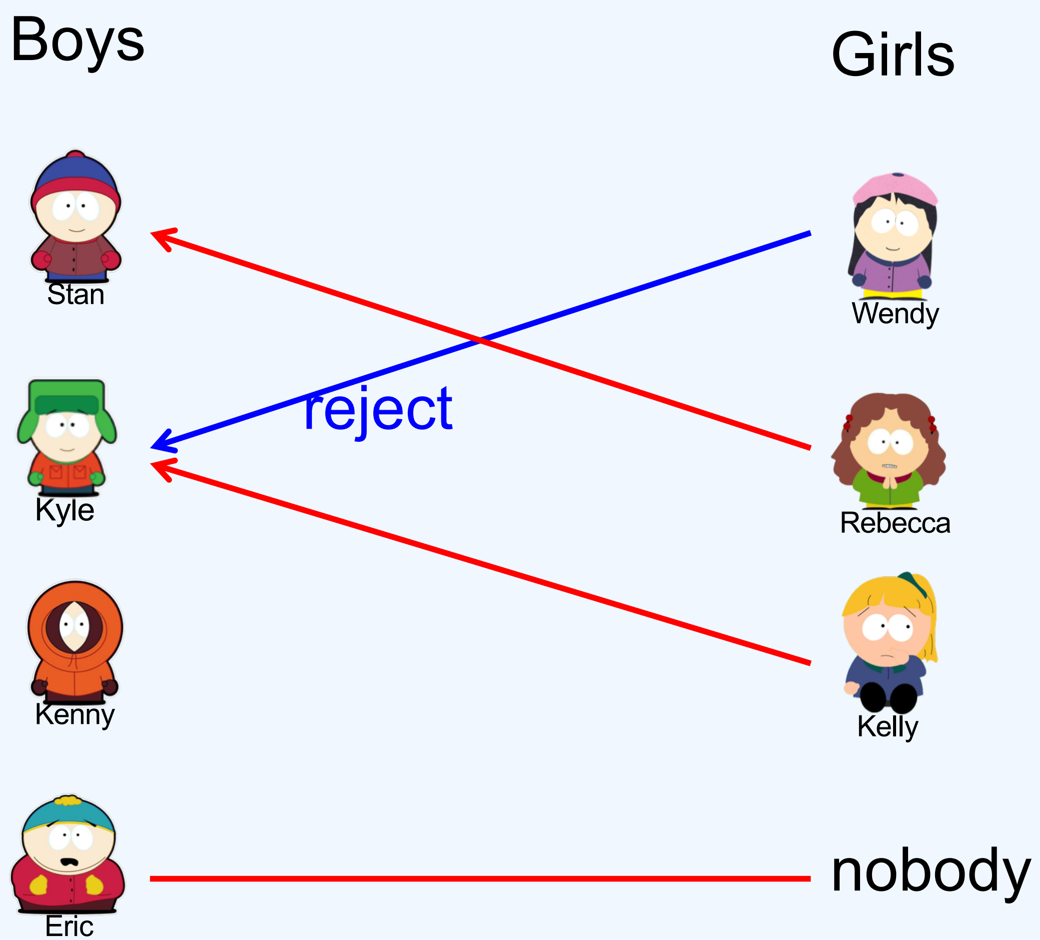
Round 2



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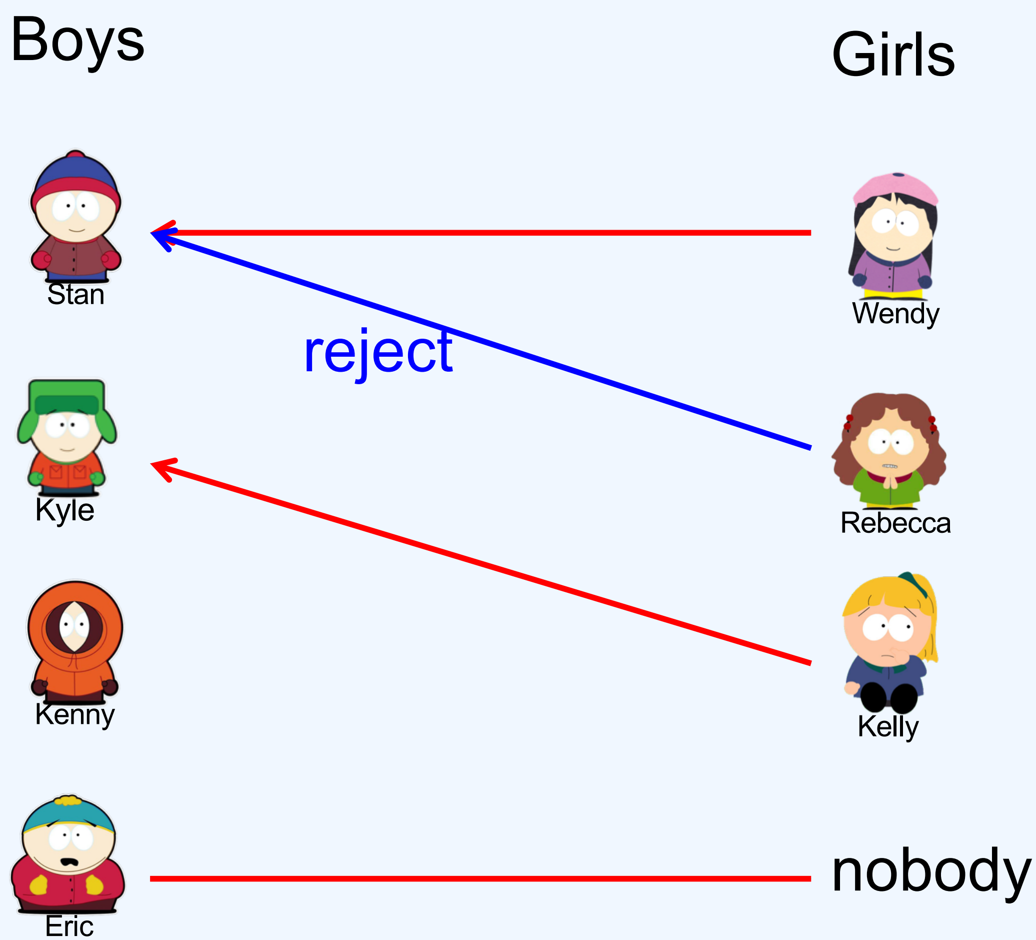
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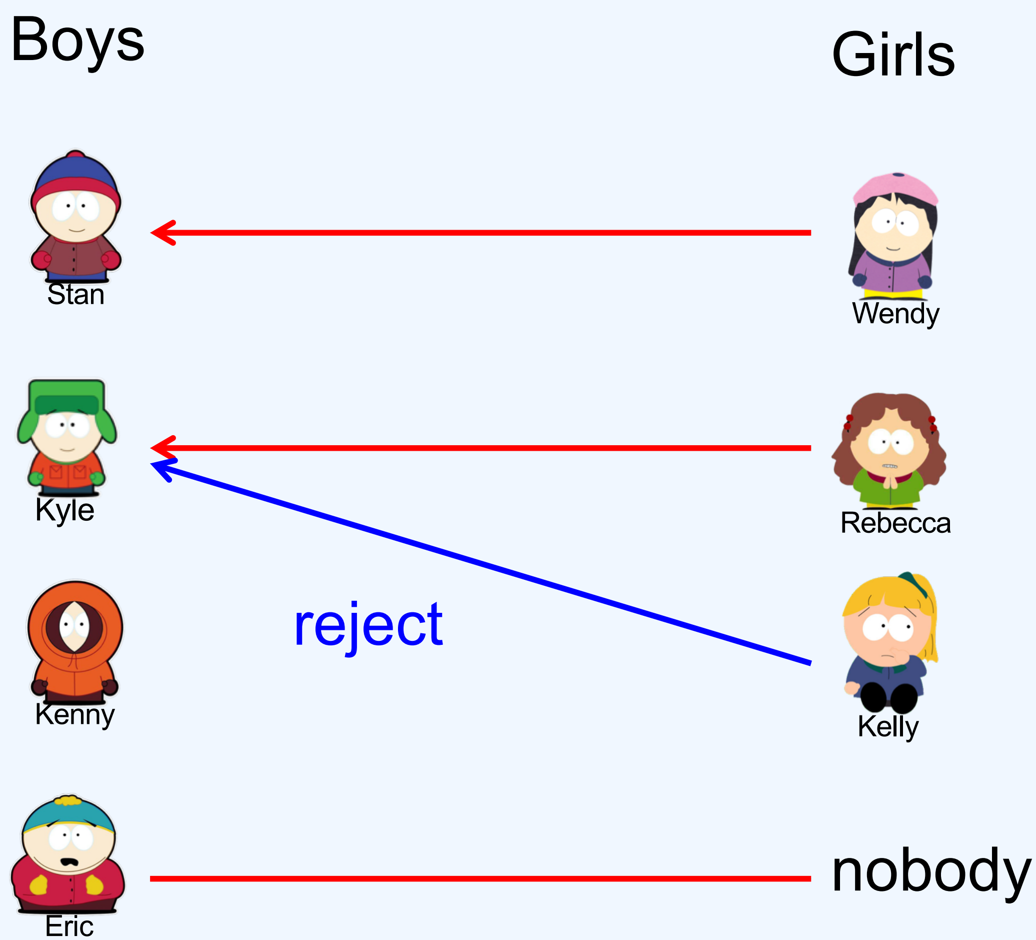
Kelly
Eric
nobody

Round 4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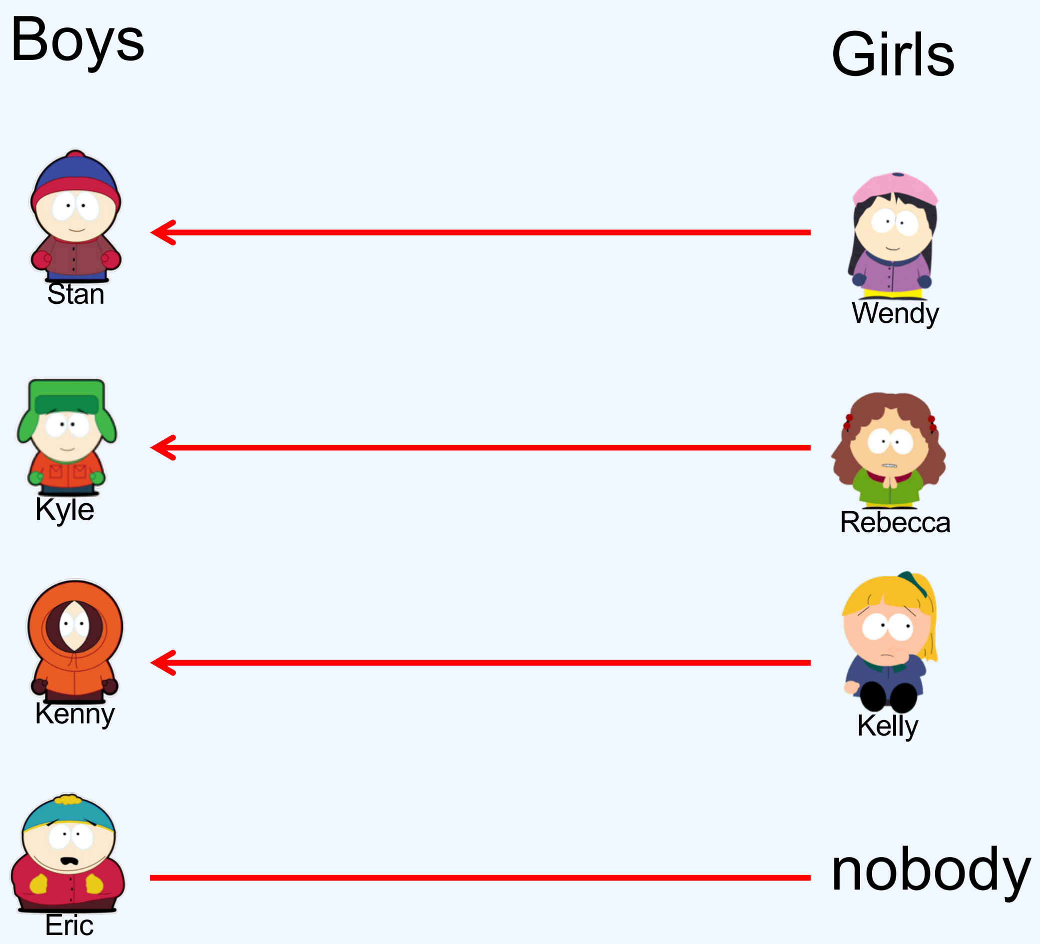
Round 5



Properties of men-proposing DA

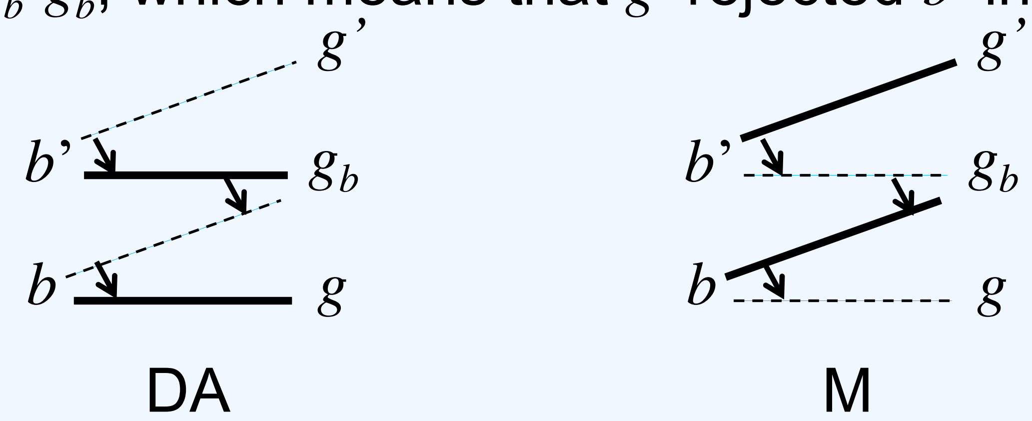
- Can be computed efficiently
- Outputs a stable matching
  - The best stable matching for boys, called men-optimal matching
  - and the worst stable matching for girls
- Strategy-proof for boys

The men-optimal matching

- For each boy b, let g_b denote his most favorable girl matched to him in any stable matching
- A matching is men-optimal if each boy b is matched to g_b
- Seems too strong, but…

Men-proposing DA is men-optimal

- Theorem. The output of men-proposing DA is men-optimal
- Proof: by contradiction
  - suppose b is the first boy not matched to $g \neq g_b$ in the execution of DA,
  - let M be an arbitrary matching where b is matched to g_b
  - Suppose b' is the boy whom g_b chose to reject b, and M(b')=g'
  - $g' >_{b'} g_b$, which means that g' rejected b' in a previous round

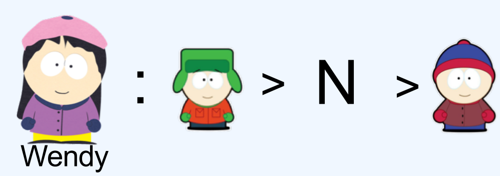


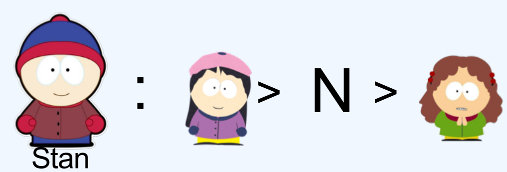
Strategy-proofness for boys

- Theorem. Truth-reporting is a dominant strategy for boys in men-proposing DA

No matching mechanism is strategy-proof and stable

- Proof.

- If (S,W) and (K,R) then 

- If (S,R) and (K,W) then 

Recap: two-sided 1-1 matching

- Men-proposing deferred acceptance algorithm (DA)
  - outputs the men-optimal stable matching
  - runs in polynomial time
  - strategy-proof on men’s side

Example

Agents

Stan

Kyle

Eric

Houses

Formal setting

- Agents $A = \{1,\ldots,n\}$
- Goods G: finite or infinite
- Preferences: represented by utility functions
  - agent j, $u_j : G \rightarrow \mathrm{R}$
- Outcomes = Allocations
  - $g : G \rightarrow A$
  - $g^{-1}: A \rightarrow 2^G$
- Difference with matching in the last class
  - 1-1 vs 1-many
  - Goods do not have preferences

Efficiency criteria

- Pareto dominance: an allocation g Pareto dominates another allocation g', if
    - all agents are not worse off under g
    - some agents are strictly better off
- Pareto optimality
  – allocations that are not Pareto dominated
- Maximizes social welfare
  – utilitarian
  – egalitarian

Fairness criteria

- Given an allocation g, agent j_1 envies agent j_2 if $u_{j_1}(g^{-1}(j_2)) > u_{j_1}(g^{-1}(j_1))$
- An allocation satisfies envy-freeness, if
  - no agent envies another agent
  - c.f. stable matching
- An allocation satisfies proportionality, if
  - for all j, $u_j(g^{-1}(j)) \geq u_j(G)/n$
- Envy-freeness implies proportionality
  - proportionality does not imply envy-freeness

Why not…

- Consider fairness in other social choice problems
  - voting: does not apply
  - matching: when all agents have the same preferences
  - auction: satisfied by the 2^{nd} price auction
- Use the agent-proposing DA in resource allocation (creating random preferences for the goods)
  - stableness is no longer necessary
  - sometimes not 1-1
  - for 1-1 cases, other mechanisms may have better properties

Allocation of indivisible goods

- House allocation
  - 1 agent 1 good
- Housing market
  - 1 agent 1 good
  - each agent originally owns a good
- 1 agent multiple goods (not discussed)

House allocation

- The same as two sided 1-1 matching except that the houses do not have preferences
- The serial dictatorship (SD) mechanism
  - given an order over the agents, w.l.o.g. $a_1 \rightarrow \ldots \rightarrow a_n$
  - in step j, let agent j choose her favorite good that is still available
  - can be either centralized or distributed
  - computation is easy

Characterization of SD

- Theorem. Serial dictatorships are the only deterministic mechanisms that satisfy
  - strategy-proofness
  - Pareto optimality
  - neutrality
  - non-bossy
    - An agent cannot change the assignment selected by a mechanism by changing his report without changing his own assigned item
- Random serial dictatorship

Why not agent-proposing DA

- Agent-proposing DA satisfies
  - strategy-proofness
  - Pareto optimality
- May fail neutrality


Stan

: h1>h2   h1: S>K


Kyle

: h1>h2   h2: K>S

- How about non-bossy?
  - No
- Agent-proposing DA when all goods have the same preferences = serial dictatorship

Housing market

- Agent j initially owns h_j
- Agents cannot misreport h_j, but can misreport her preferences
- A mechanism f satisfies participation
  - if no agent j prefers h_j to her currently assigned item
- An assignment is in the core
  - if no subset of agents can do better by trading the goods that they own in the beginning among themselves
  - stronger than Pareto-optimality

Example: core allocation

: h1>h2>h3, owns h3

: h3>h2>h1, owns h1

: h3>h1>h2, owns h2

Not in the core

: h2

: h3

: h1

In the core

: h1

: h3

: h2

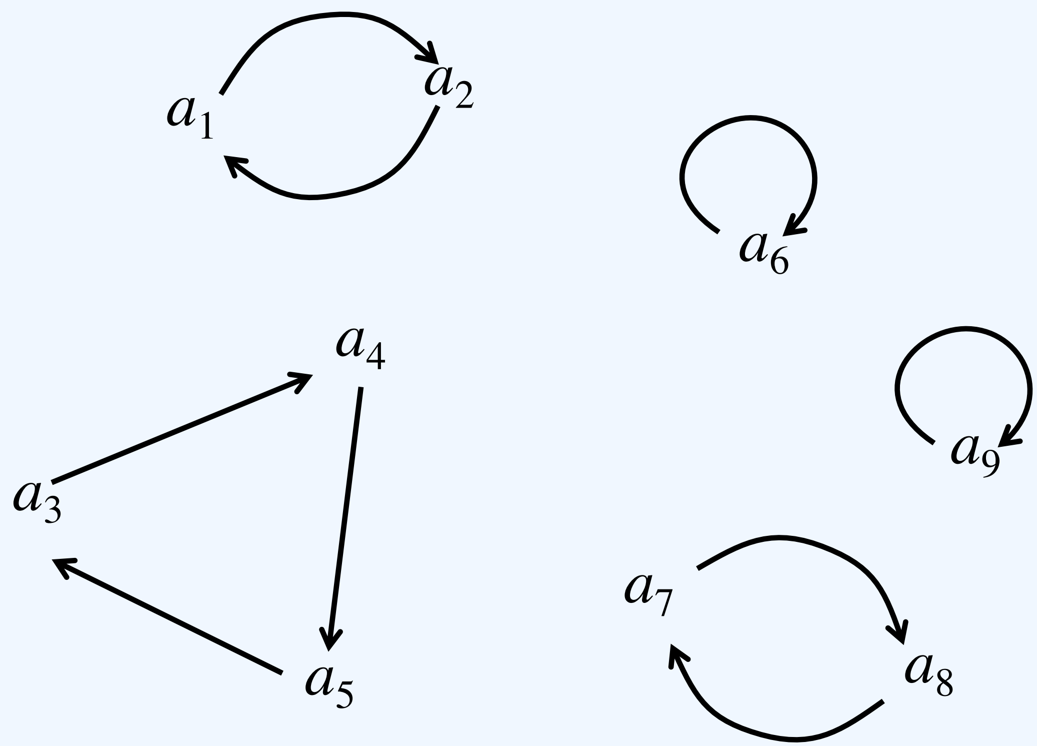
The top trading cycles (TTC) mechanism

- Start with: agent j owns h_j
- In each round
  - built a graph where there is an edge from each available agent to the owner of her most-preferred house
  - identify all cycles; in each cycle, let the agent j gets the house of the next agent in the cycle; these will be their final allocation
  - remove all agents in these cycles

Example

a_1: h_2>… a_2: h_1>… a_3: h_4>… a_4: h_5>… a_5: h_3>… a_6: h_4>h_3>h_6>…

a_7: h_4>h_5>h_6>h_3>h_8>… a_8: h_7>… a_9: h_6>h_4>h_7>h_3>h_9>…



Properties of TTC

- Theorem. The TTC mechanism
  - is strategy-proof
  - is Pareto optimal
  - satisfies participation
  - selects an assignment in the core
    - the core has a unique assignment
  - can be computed in $O(n^2)$ time
- Why not using TTC in 1-1 matching?
  - not stable
- Why not using TTC in house allocation (using random initial allocation)?
  - not neutral

DA vs SD vs TTC

- All satisfy
  - strategy-proofness
  - Pareto optimality
  - easy-to-compute
- DA
  - stableness
- SD
  - neutrality
- TTC
  - chooses the core assignment

Multi-type resource allocation

- Each good is characterized by multiple issues
  - e.g. each presentation is characterized by topic and time
- Paper allocation
  - we have used SD to allocate the topic
  - we will use SD with reverse order for time
- Potential research project