Auctions

Lirong Xia

Rensselaer

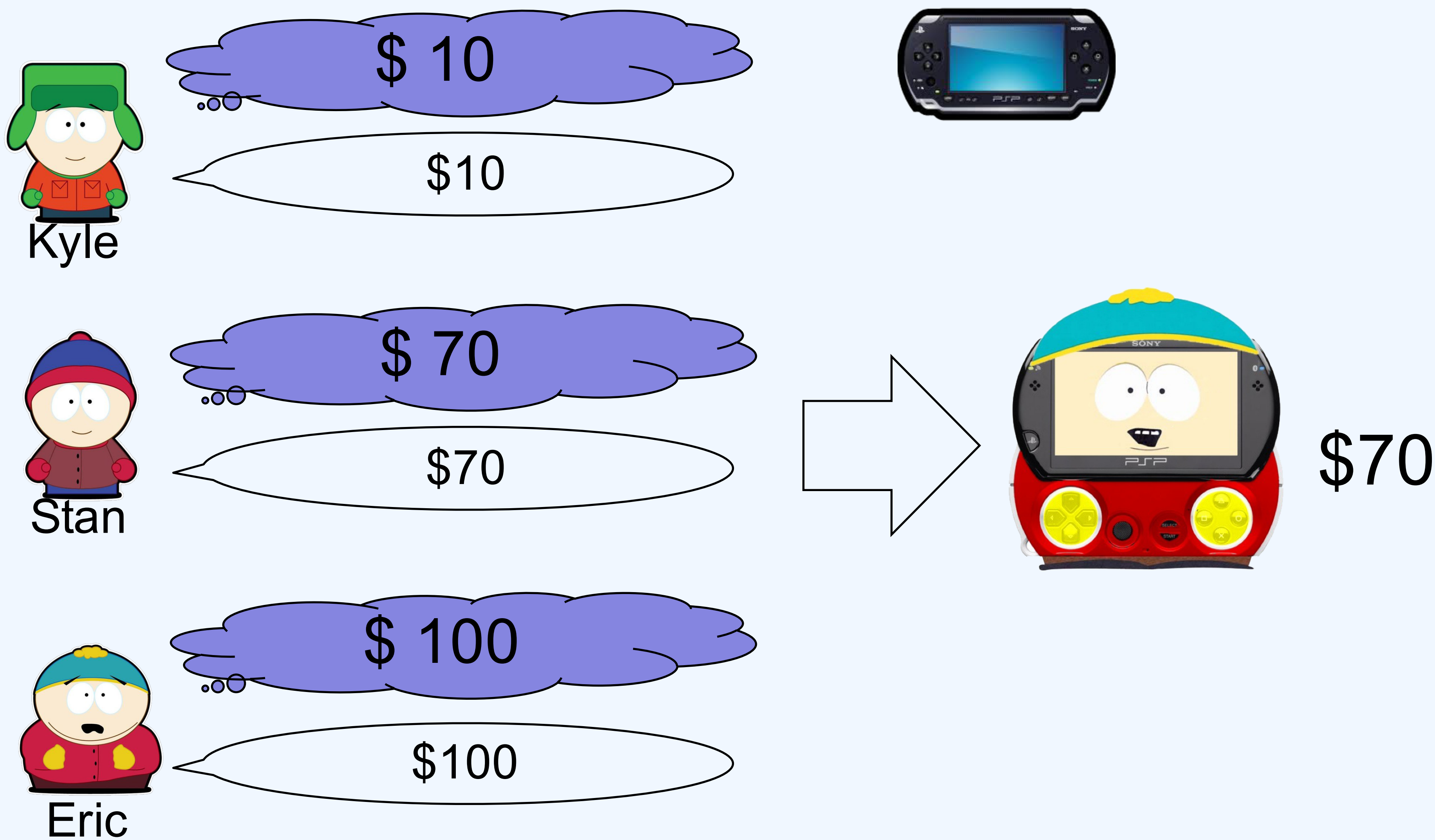
Sealed-Bid Auction

- One item
- A set of bidders $1,\ldots,n$
  - bidder j's true value v_j
  - bid profile $b = (b_1,\ldots,b_n)$
- A sealed-bid auction has two parts
  - allocation rule: $x(b) \in \{0,1\}^n$, $x_j(b)=1$ means agent j gets the item
  - payment rule: $p(b) \in R^n$, $p_j(b)$ is the payment of agent j
- Preferences: quasi-linear utility function
  - $x_j(b)\, v_j - p_j(b)$

Second-Price Sealed-Bid Auction

- W.l.o.g. $b_1 \geq b_2 \geq \ldots \geq b_n$
- Second-Price Sealed-Bid Auction
  - $x_{SP}(b) = (1,0,\ldots,0)$ (item given to the highest bid)
  - $p_{SP}(b) = (b_2,0,\ldots,0)$ (charged 2nd highest price)

Example



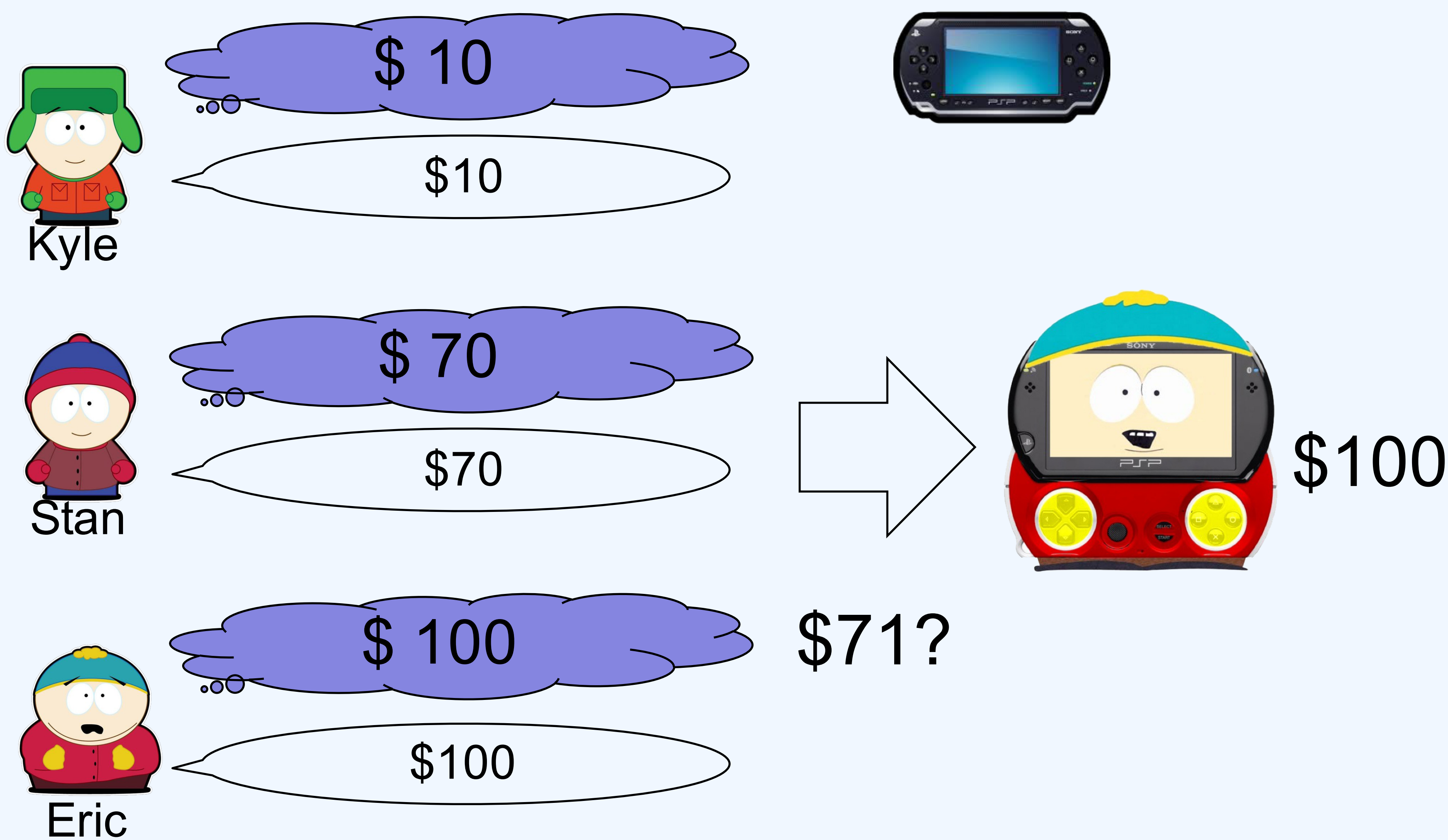
Incentive Compatibility of 2nd Price Auction

- Dominant-strategy Incentive Compatibility (DSIC)
  - reporting true value is the best regardless of other agents' actions
- Why?
  - underbid ($b \leq v$)
    - win → win: no difference
    - win → lose: utility = 0 ≤ truthful bidding
  - overbid ($b \geq v$)
    - win → win: no difference
    - lose → win: utility ≤ 0 ≤ truthful bidding
- Nash Equilibrium
  - everyone bids truthfully

First-Price Sealed-Bid Auction

- W.l.o.g. $b_1 \geq b_2 \geq \ldots \geq b_n$
- First-Price Sealed-Bid Auction
  - $x_{FP}(b)$ = (1,0,…,0) (item given to the highest bid)
  - $p_{FP}(b)$ = (b_1,0,…,0) (charged her reported price)

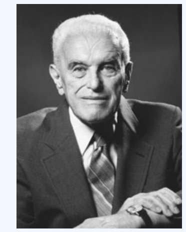
Example



Nash Equilibrium of 1^{st} Price Auction

- Complete information
  - max bid = 2^{nd} bid + ε
- Not sure about other bidders' values?
  - winner's curse

Games of Incomplete Information for auctions



Harsanyi

- Bidder j's type = her value θ_j (private)
  - quasi-linear utility functions
- G: joint distribution of bidders' (true) values (public)
- Strategy: s_j : R → R (from type to bid)
- Timing

Ex ante

  - 1. Generate $(\theta_1,\ldots,\theta_n)$ from G, bidder j receives θ_j

Interim

  - 2. Bidder j reports $s_j(\theta_j)$

Ex post

  - 3. Allocation and payments are announced

Bayes-Nash Equilibrium

- A strategy profile $(s_1,\ldots,s_n)$ is a Bayes-Nash Equilibrium (BNE) if for every agent j, all types θ_j, and all potential deviations b_j', we have

other agents' bids　　　unilateral deviation

$$E_{\theta_{-j}}\, u_j(s_j(\theta_j),\ s_{-j}(\theta_{-j}) \mid \theta_j) \geq E_{\theta_{-j}}\, u_j(b_j',\ s_{-j}(\theta_{-j}) \mid \theta_j)$$

your bids　　　conditioned on j's information

- $s_{-j} = (s_1,\ldots,s_{j-1},\ s_{j+1},\ldots,s_n)$

BNE of 1st Price Auction

- Proposition. When all values are generated i.i.d. from uniform[0,1], under 1st price auction, the strategy profile where for all j, s_j: $\theta \rightarrow \frac{n-1}{n}\theta$ is a BNE
- Proof.
  - suppose bidder j's value is θ_j and she decides to bid for $b_j \leq \theta_j$
  - Expected payoff

$(\theta_j - b_j) \times \Pr(b_j \text{ is the highest bid})$

$= (\theta_j - b_j) \times \Pr(\text{all other bids} \leq b_j \mid s_{-j})$

$= (\theta_j - b_j) \times \Pr(\text{all other values} \leq \frac{n}{n-1} b_j)$

$= (\theta_j - b_j)(\frac{n}{n-1} b_j)^{n-1}$

  - maximized at $b_j = \frac{n-1}{n}\theta_j$

BNE of 2nd Price Auction

- $b_j = \theta_j$
- Dominant-Strategy Incentive Compatibility

Desirable Auctions

- Efficiency in equilibrium (allocate the item to the agent with the highest value)
  - 🙂 1st price auction
  - 🙂 2nd price auction
- Revenue in equilibrium

Expected Revenue in Equilibrium: 1st price auction

- Expected revenue for 1st price auctions with i.i.d. Uniform[0,1] when $b_j = \frac{n-1}{n} v_j$

$$\int_0^{\frac{n-1}{n}} b \times \Pr(\text{highest bid is } b)\, d\theta$$

$$= \int_0^1 \frac{n-1}{n}\theta \times \Pr(\text{highest value is } \theta)\, d\theta$$

$$= \int_0^1 \frac{n-1}{n}\theta \times n\theta^{n-1}\, d\theta$$

$$= \frac{n-1}{n+1}$$

Expected Revenue in Equilibrium: 2^{st} price auction

- Expected revenue for 2^{st} price auctions with i.i.d. Uniform[0,1] when $b_j = v_j$

$$\int_0^1 b \times \Pr(2^{\text{nd}} \text{ highest bid is } b)\, db$$

$$= n(n-1)\int_0^1 \theta \times (1-\theta)\theta^{n-2} d\theta$$

$$= n(n-1)\int_0^1 \theta^{n-1} - \theta^n d\theta$$

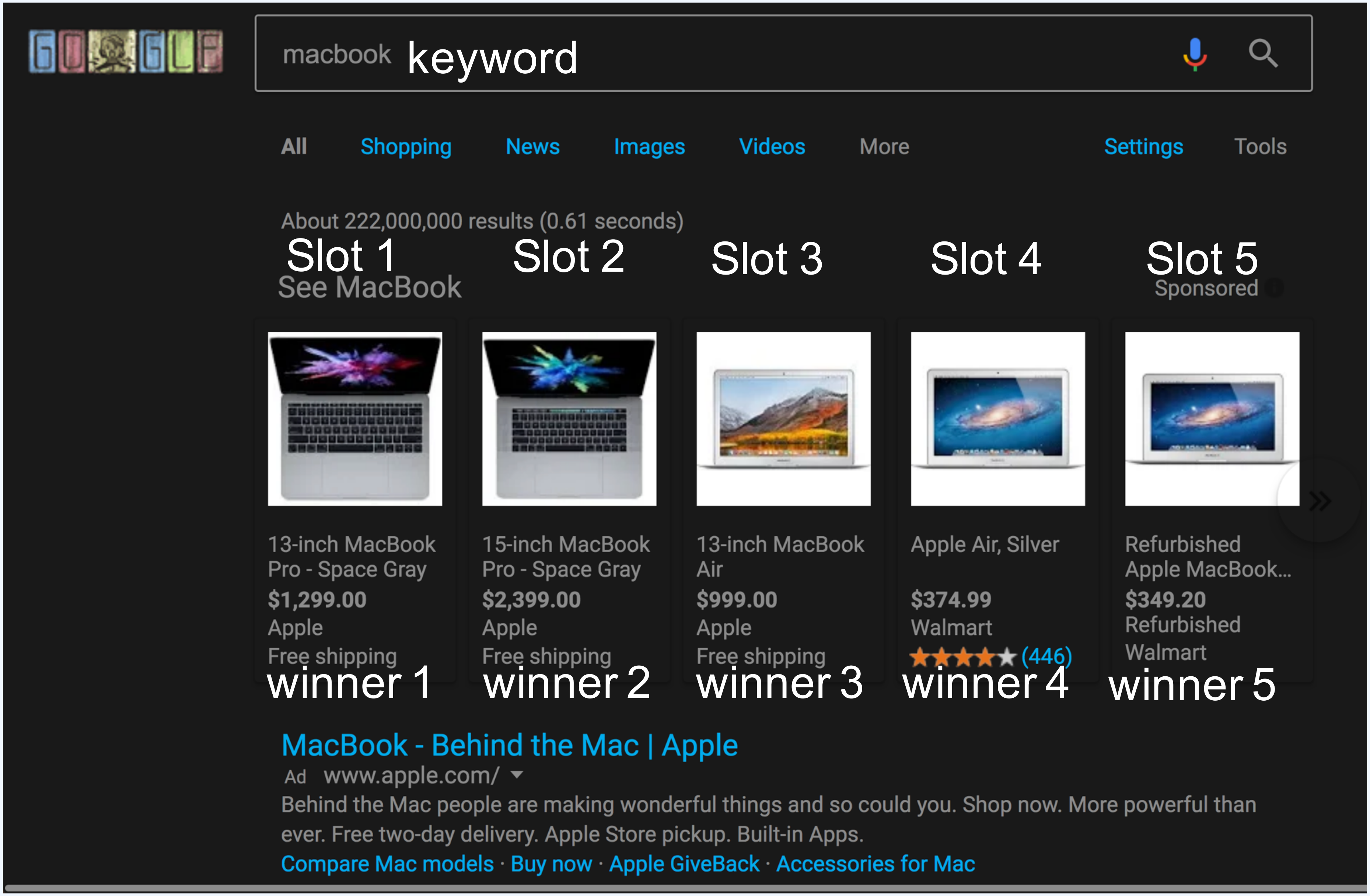
$$= \frac{n-1}{n+1}$$

= expected revenue of 1^{st} price auction in equilibrium

A Revenue Equivalence Theorem

- Theorem. The expected revenue of all auction mechanisms for a single item satisfying the following conditions are the same
  - highest bid wins the items (break ties arbitrarily)
  - there exists an BNE where
    - symmetric: all bidders use the same strategy
      - does not mean that they have the same type
    - increasing: bid increase with the value
- Example: 1^{st} price vs. 2^{nd} price auction

Ad Auction


macbook keyword
All
Shopping
News
Images
Videos
More
Settings
Tools
About 222,000,000 results (0.61 seconds)
Slot 1
See MacBook
Slot 2
Slot 3
Slot 4
Slot 5
Sponsored
13-inch MacBook Pro - Space Gray
$1,299.00
Apple
Free shipping
winner 1
15-inch MacBook Pro - Space Gray
$2,399.00
Apple
Free shipping
winner 2
13-inch MacBook Air
$999.00
Apple
Free shipping
winner 3
Apple Air, Silver
$374.99
Walmart
(446)
winner 4
Refurbished Apple MacBook...
$349.20
Refurbished
Walmart
winner 5
MacBook - Behind the Mac | Apple
Ad www.apple.com/
Behind the Mac people are making wonderful things and so could you. Shop now. More powerful than ever. Free two-day delivery. Apple Store pickup. Built-in Apps.
Compare Mac models · Buy now · Apple GiveBack · Accessories for Mac

Ad Auctions: Setup

- m slots
  - slot i gets s_i clicks
- n bidders
  - v_j : value for each user click
  - b_j : pay (to service provider) per click
  - utility of getting slot i : $(v_j - b_j) \times s_i$
- Outcomes: { (allocation, payment) }

Generalized 2nd price Auction (GSP)

- Rank the bids
  - W.l.o.g. $b_1 \geq b_2 \geq \ldots \geq b_n$
- for $i = 1$ to m,
  - give slot i to b_i
  - charge bidder i to b_{i+1} pay per click
- Example
  - $n=4, m=3; s_1 = 100, s_2 = 60, s_3 = 40; v_1 = 10, v_2 = 9, v_3 = 7, v_4 = 1.$
  - bidder 1 utility
  - HW: show GSP is not incentive compatible