Economics and Computation
Homework 3

Please do not hand in. Just do it by yourself to prepare for the exam.

1 Highly Recommended

Problem 1. ILP for Kemeny (2pts) Prove that it suffices to check conditions for all cycles of length 3 in the ILP of Kemeny. I.e. if x_{ab}'s do not correspond to a linear order then one of the constraints must be violated.

Problem 2. Manipulation (8pts) Suppose agent 1 is a manipulator, whose preferences are $a \succ b \succ c \succ d$. Let P_{-1} denote the votes of other agents.

$$P_{-1} = 1@[b \succ c \succ d \succ a] + 2@[c \succ d \succ a \succ b] + 1@[d \succ a \succ b \succ c]$$

For each of the following rule, either show a beneficial manipulation of agent 1, or say that agent 1 has no incentive to manipulate (no proof is needed for the latter). Notice that **a beneficial manipulation means that agent 1 must prefers the new winner to the old one w.r.t. to her true preferences $a \succ b \succ c \succ d$.** Ties are broken alphabetically in favor of alternatives with higher priority (for example, when eliminating tied alternatives, those with lower priority will be eliminated).

1. Plurality.

2. Borda.

3. Veto.
4. Plurality with runoff.
5. STV.
6. Copeland.
7. Ranked Pairs.

Problem 3. Single-Peaked Preferences (3pts)

1. Is $b \succ c \succ a \succ d$ consistent with the social axis $a \rhd b \rhd c \rhd d$? (The best way to verify this is to draw the plot as we did in the class).
2. Is $a \succ c \succ d \succ b$ consistent with the social axis $a \rhd b \rhd c \rhd d$?
3. Prove that if a linear order V is consistent with a social axis S if and only if it is consistent with the reserves ranking $rev(S)$. For example, if $S = a \rhd b \rhd c$ then $rev(S) = c \rhd b \rhd a$. **Your proof should work for all S, not just this example. Notice that it must be a formal proof by verifying the definition of single-peakedness. Drawing the plot is not a valid formal proof.**

2 Food for Thought

Problem 4. (3pt) Prove that for single-peaked preferences, any median rule with phantom voters is strategy-proof.

Problem 5. (3pt) Let the voting rule be STV.

1. Consider the following profile:

$$27@[a \succ b \succ c] \qquad 42@[c \succ a \succ b] \qquad 24@[b \succ c \succ a]$$

   What happens when four votes switch from $a \succ b \succ c$ to $c \succ a \succ b$, and what axiomatic property does this violate?

2. For the same profile in (a), what paradoxical outcome occurs when four voters with $a \succ b \succ c$ don't vote?

3. Prove that STV does *not* satisfy consistency.

Problem 6. (3pt) Prove that all positional-scoring rules satisfy consistency. You can assume that there are no ties in the profiles.

Problem 7. (2pt) Prove that for any profile P, let $WMG(P)$ denote the weighted majority graph. Prove that one of the following two cases must hold: (1) weights on all edges of in $WMG(P)$ are even numbers; or (2) weights on all edges of in $WMG(P)$ are odd numbers.

Problem 8. Bonus question: (5pt) Let $\vec{s}_B = (m-1, \ldots, 0)$ denote the scoring vector for Borda.

1. Prove that for any $p > 0, q \in \mathbb{R}$, the positional scoring rule r with the scoring vector $p \cdot \vec{s}_B + q = (p(m-1)+q, p(m-2)+q, \ldots, q)$ is equivalent to Borda. That is, for any profile P, $r(P) = \text{Borda}(P)$.

2. Prove the reverse of (a). That is, prove that a position scoring rule r with scoring vector $\vec{s} = (s_1, \ldots, s_m)$ is equivalent to Borda only if there exist $p > 0, q \in \mathbb{R}$ such that $\vec{s} = p \cdot \vec{s}_B + q$.
   Hint: show that $s_1 - s_2 = s_2 - s_3 = \cdots = s_{m-1} - s_m$.

Problem 9. bonus question (5pts) Given $m \in \mathbb{N}$ and m positional scoring rules with scoring vectors $(\vec{s^1}, \vec{s^2}, \ldots, \vec{s^m})$, where $\vec{s^i} = (s^i_1, \ldots, s^i_m)$.

Design a mixed integer programming to find a profile P with the smallest number of votes so that all these m positional scoring rules output different winners. **Your ILP should use only polynomially many (in m and n) variables and constraints.**
Remarks: The ILP should be able to identify "failures", that is, situations where such a profile does not exist. You don't need to write down all constraints explicitly, but make sure that every parameter you use in the ILP is well defined.
Hint: Check out "Doubly stochastic matrix" and "Birkhoff–von Neumann theorem" at `http://en.wikipedia.org/wiki/Doubly_stochastic_matrix`

Problem 10. (**hard**) Prove step 2 in the proof idea of Gibbard-Satterthwaite theorem. Namely, suppose there exists a non-dictatorial and strategy-proof voting rule r, then for any profile P an any three alternatives a, b, c, we must have that if $r(P_{ab}) = a$ and $r(P_{bc}) = b$, then $r(P_{ac}) = a$.