CS 534: Computer Vision
Camera Geometry

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Outlines

- Projective Geometry
- Homogenous coordinates
- Euclidean Geometry
- Rigid Transformations
- Perspective Projection
- Other projection models
  - Weak perspective projection
  - Orthographic projection
- Camera intrinsic and extrinsic parameters

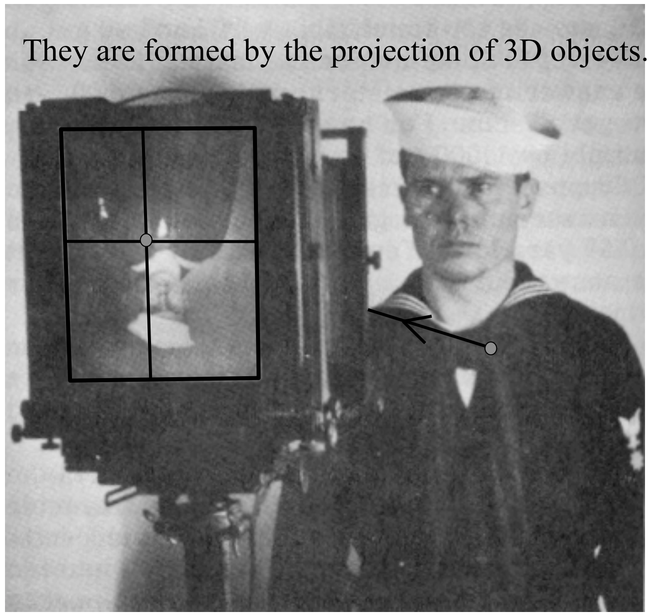


Figure from US Navy Manual of Basic Optics and Optical Instruments, prepared by Bureau of Naval Personnel. Reprinted by Dover Publications, Inc., 1969.

Images are two-dimensional patterns of brightness values.

Images of the 3-D world

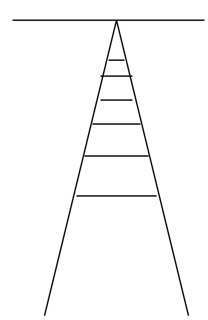
- What is the geometry of the image of a three-dimensional object?
  - Given a point in space, where will we see it in an image?
  - Given a line segment in space, what does its image look like?
  - Why do the images of lines that are parallel in space appear to converge to a single point in an image?
- How can we recover information about the 3-D world from a 2-D image?
  - Given a point in an image, what can we say about the location of the 3-D point in space?
  - Are there advantages to having more than one image in recovering 3-D information?
  - If we know the geometry of a 3-D object, can we locate it in space (say for a robot to pick it up) from a 2-D image?

Projective geometry 101

- Euclidean geometry describes shapes "as they are"
  - properties of objects that are unchanged by rigid motions
    - lengths
    - angles
    - parallelism
- Projective geometry describes objects "as they appear"
  - lengths, angles, parallelism become "distorted" when we look at objects
  - mathematical model for how images of the 3D world are formed

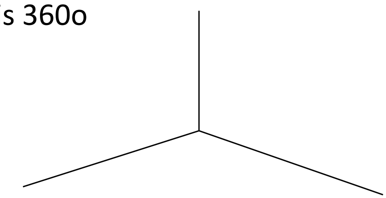
Example 1

- Consider a set of railroad tracks
  - Their actual shape:
    - tracks are parallel
    - ties are perpendicular to the tracks
    - ties are evenly spaced along the tracks
  - Their appearance
    - tracks converge to a point on the horizon
    - tracks don't meet ties at right angles
    - ties become closer and closer towards the horizon



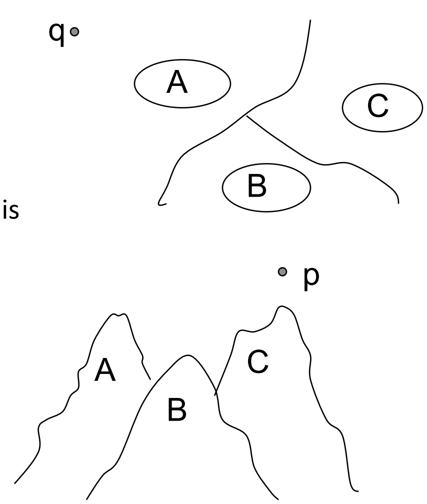
Example 2

- Corner of a room
  - Actual shape
    - three walls meeting at right angles. Total of 270o of angle.
  - Appearance
    - a point on which three lines segments are concurrent. Total angle is 360o



Example 3

- B appears between A and C from point p
- But from point q, A appears between B and C
- Apparent displacement of objects due to change in viewing position is called parallax shift



Projective Geometry

- Non-metrical description
- Less restrictive/ more general than Euclidean
- Describes properties that are invariant under projective transformation
- Invariant properties include
  - Incidence
  - Cross Ratio

Projective Geometry

- Classical Euclidean geometry: through any point not on a given line, there exists a unique line which is parallel to the given line.
  - For 2,000 years, mathematician tried to “prove” this from Euclid’s postulates.
  - In the early 19’th century, geometry was revolutionized when mathematicians asked: What if this were false?
  - That is, what if we assumed that EVERY pair of lines intersected?
  - To do this, we’ll have to add points and lines to the standard Euclidean plane.

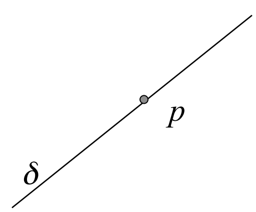
Homogeneous coordinates

- If (x,y) are the rectangular coordinates of a point, P, and if (x1, x2, x3) are any three real numbers such that:
  - x1/x3 = x
  - x2/x3 = y
- then (x1, x2, x3) are a set of homogeneous coordinates for (x,y).
- So, in particular, (x,y,1) are a set of homogeneous coordinates for (x,y)
- Given the homogeneous coordinates, (x1, x2, x3), the rectangular coordinates can be recovered.
- But (x,y) has an infinite number of homogeneous coordinate representations, because if (x1, x2, x3) are homogeneous coordinates of (x,y), then so are (kx1, kx2,k x3) for any k <>0.

Homogenous Coordinates Lines

- On 2D plane

$$ax + by - d = 0 \Leftrightarrow \delta^T \cdot p = 0$$



$$\delta = \begin{pmatrix} a \\ b \\ -d \end{pmatrix} \qquad p = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Homogenous Coordinates

- Intersection of two lines

$$x = l \times l'$$
$$proof : l \cdot (l \times l') = 0, l' \cdot (l \times l') = 0$$

- Intersection of parallel lines ?
- Line passing through two points: $p_1 x p_2$
- Three points on the same line: det[p_1 p_2 p_3]=0
- Three line intersecting in a point: det[l_1 l_2 l_3] =0

point	$\mathbf{p} = (X, Y, W)$
incidence	$\mathbf{p}^T\mathbf{u} = 0$
collinearity	$\lvert \mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3 \rvert = 0$
join of 2 points	$\mathbf{u} = \mathbf{p}_1 \times \mathbf{p}_2$
ideal points	$(X, Y, 0)$

(a)

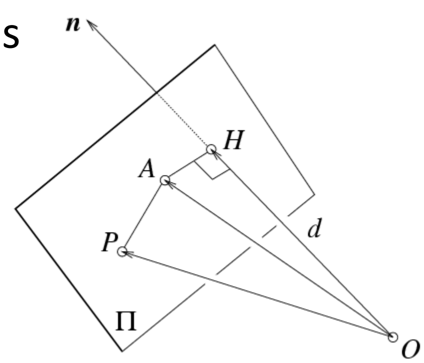
line	$\mathbf{u} = (a, b, c)$
incidence	$\mathbf{p}^T\mathbf{u} = 0$
concurrence	$\lvert \mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3 \rvert = 0$
intersection of 2 lines	$\mathbf{p} = \mathbf{u}_1 \times \mathbf{u}_2$
ideal line	$(0, 0, c)$

(b)

Table from S. Birchfield "An Introduction to Projective Geometry"

Homogenous Coordinates
Planes

$$P = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad n = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$



$$\overrightarrow{AP}.\mathbf{n} = 0 \Leftrightarrow OP\cdot n - OA\cdot n = 0 \Leftrightarrow ax + by + cz - d = 0 \Leftrightarrow \mathbf{\Pi}^{\mathrm{T}}.\mathbf{P} = 0$$

where $\mathbf{\Pi} = \begin{bmatrix} a \\ b \\ c \\ -d \end{bmatrix}$ and $\mathbf{P} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$ Homogenous coordinates

- Represent both points and lines as 4 numbers
- For 3D points, just add one to obtain its homogenous coordinate.
- Homogenous coordinates are defined up to scale: multiplying by any nonzero scale will not change the equation:

$$\begin{pmatrix} a & b & c & -d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = 0$$

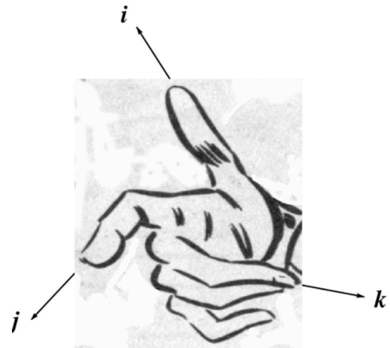
Homogenous Coordinates

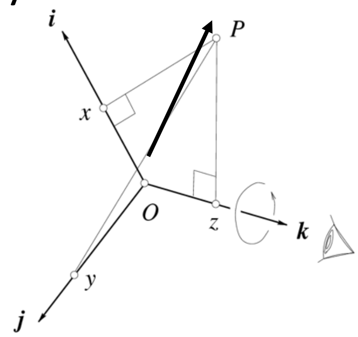
- Spheres

$$x^2 + y^2 + z^2 = R^2$$

$$\begin{pmatrix} x & y & z & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & R^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = 0$$

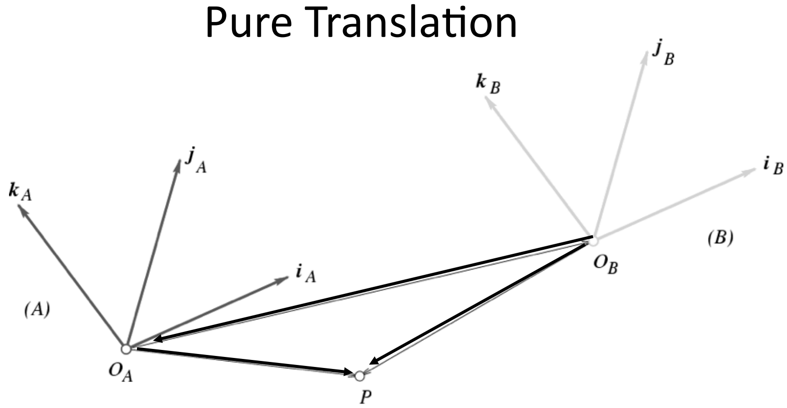
Euclidean Geometry
Coordinate Systems





$$\begin{cases} x = \overrightarrow{OP}.\mathbf{i} \\ y = \overrightarrow{OP}.\mathbf{j} \\ z = \overrightarrow{OP}.\mathbf{k} \end{cases} \Leftrightarrow \quad \overrightarrow{OP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \quad \Leftrightarrow \quad \mathbf{P} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Coordinate System Change
Pure Translation



$$\overrightarrow{O_B P} = \overrightarrow{O_B O_A} + \overrightarrow{O_A P} \;\; , \;\; {}^B P = {}^A P + {}^B O_A$$

Coordinate System Change
Pure Translation

Notations:
Left Superscript : Coordinate Frame of Reference

Origin of coordinate frame A in coordinate frame B

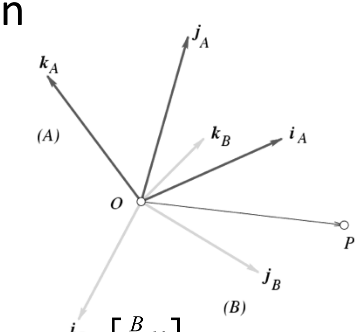
Point P in coordinate Frame A

Point P in coordinate Frame B

$$\overrightarrow{O_BP} = \overrightarrow{O_BO_A} + \overrightarrow{O_AP}\ ,\ \ {}^{B}P = {}^{A}P + {}^{B}O_A$$

Coordinate System Change
Pure Rotation

Multiply both sides by $\begin{bmatrix} i_B \\ j_B \\ k_B \end{bmatrix}$

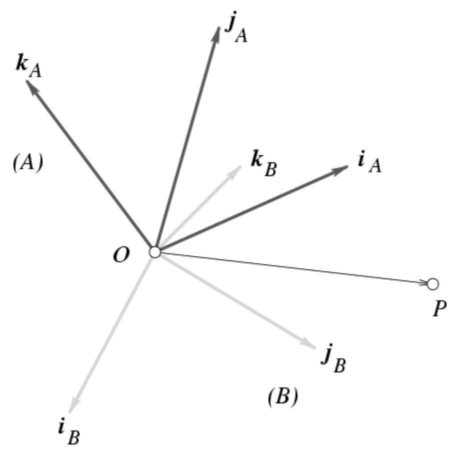


$$\overrightarrow{OP} = \begin{bmatrix} \mathbf{i}_A & \mathbf{j}_A & \mathbf{k}_A \end{bmatrix} \begin{bmatrix} {}^{A}x \\ {}^{A}y \\ {}^{A}z \end{bmatrix} = \begin{bmatrix} \mathbf{i}_B & \mathbf{j}_B & \mathbf{k}_B \end{bmatrix} \begin{bmatrix} {}^{B}x \\ {}^{B}y \\ {}^{B}z \end{bmatrix}$$

$$\Rightarrow \quad {}^{B}P = {}^{B}_{A}R\,{}^{A}P$$

$${}^{B}_{A}R = \begin{bmatrix} \mathbf{i}_A.\mathbf{i}_B & \mathbf{j}_A.\mathbf{i}_B & \mathbf{k}_A.\mathbf{i}_B \\ \mathbf{i}_A.\mathbf{j}_B & \mathbf{j}_A.\mathbf{j}_B & \mathbf{k}_A.\mathbf{j}_B \\ \mathbf{i}_A.\mathbf{k}_B & \mathbf{j}_A.\mathbf{k}_B & \mathbf{k}_A.\mathbf{k}_B \end{bmatrix}$$

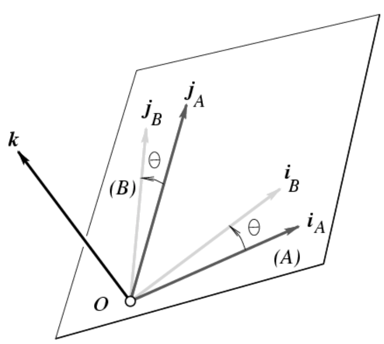
Coordinate Changes: Pure Rotations



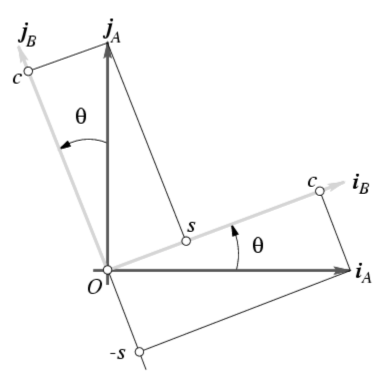
Rotation matrix describing the frame A in coordinate frame B

$$ {}^{B}_{A}R = \begin{bmatrix} \mathbf{i}_A . \mathbf{i}_B & \mathbf{j}_A . \mathbf{i}_B & \mathbf{k}_A . \mathbf{i}_B \\ \mathbf{i}_A . \mathbf{j}_B & \mathbf{j}_A . \mathbf{j}_B & \mathbf{k}_A . \mathbf{j}_B \\ \mathbf{i}_A . \mathbf{k}_B & \mathbf{j}_A . \mathbf{k}_B & \mathbf{k}_A . \mathbf{k}_B \end{bmatrix} = \begin{bmatrix} {}^{A}\mathbf{i}_B^T \\ {}^{A}\mathbf{j}_B^T \\ {}^{A}\mathbf{k}_B^T \end{bmatrix} = \begin{bmatrix} {}^{B}\mathbf{i}_A & {}^{B}\mathbf{j}_A & {}^{B}\mathbf{k}_A \end{bmatrix} $$

Coordinate Changes: Rotations about the *z* Axis



$$ {}^{B}_{A}R = \begin{bmatrix} \mathbf{i}_A . \mathbf{i}_B & \mathbf{j}_A . \mathbf{i}_B & \mathbf{k}_A . \mathbf{i}_B \\ \mathbf{i}_A . \mathbf{j}_B & \mathbf{j}_A . \mathbf{j}_B & \mathbf{k}_A . \mathbf{j}_B \\ \mathbf{i}_A . \mathbf{k}_B & \mathbf{j}_A . \mathbf{k}_B & \mathbf{k}_A . \mathbf{k}_B \end{bmatrix} $$



$$ {}^{B}_{A}R = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

A rotation matrix is characterized by the following properties:

- Its inverse is equal to its transpose, and
- its determinant is equal to 1.

Or equivalently:

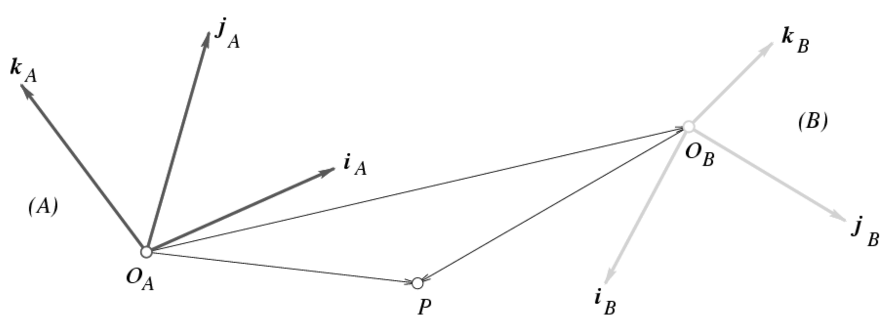
- Its rows (or columns) form a right-handed orthonormal coordinate system.

example

$$ {}^{B}_{A}R = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

- In general, any rotation matrix can be written as the product of three elementary rotations about the i, j, and k vectors

Coordinate Changes: Rigid Transformations



$$ {}^{B}P = {}^{B}_{A}R \; {}^{A}P + {}^{B}O_A $$

Block Matrix Multiplication

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

What is AB ?

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

$${}^{B}P = {}^{B}_{A}R\ {}^{A}P + {}^{B}O_{A}$$

Homogeneous Representation of Rigid Transformations

$$\boxed{\begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{B}_{A}R\ {}^{A}P + {}^{B}O_{A} \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{B}_{A}R & {}^{B}O_{A} \\ \mathbf{0}^{T} & 1 \end{bmatrix} \begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = {}^{B}_{A}T \begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix}}$$

Other transformations

$$T = \begin{pmatrix} A & t \\ 0^{T} & 1 \end{pmatrix}$$

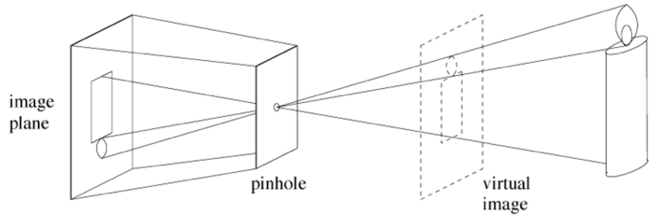
A is 3x3 rotation matrix :
Rigid transformation
Lengths and angels are preserved

A is arbitrary (nonsingular) 3x3 matrix :
Affine transformation
Lengths and angles may not be preserved

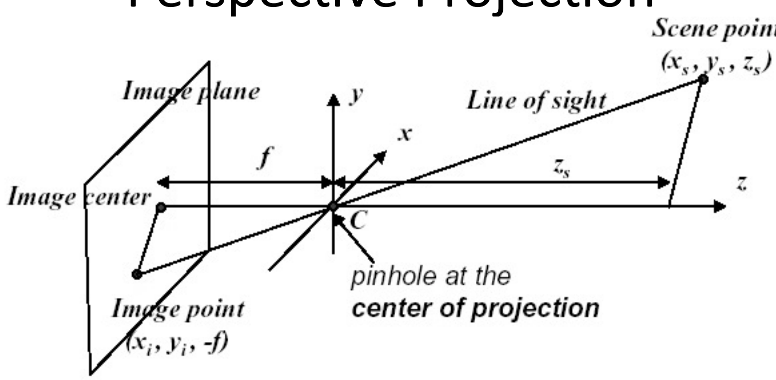
T is arbitrary (nonsingular) 4x4 matrix :
projective transformation
Lengths and angles may not be preserved

Pinhole Perspective

- Abstract camera model - box with a small hole in it
- Assume a single point pinhole (ideal pinhole):
  - Pinhole (central) perspective projection {Brunellleschi 15th Century}
  - Extremely simple model for imaging geometry
  - Doesn't strictly apply
  - Mathematically convenient – acceptable approximation.
  - Concepts: image plane, virtual image plane
  - Moving the image plane merely scales the image.

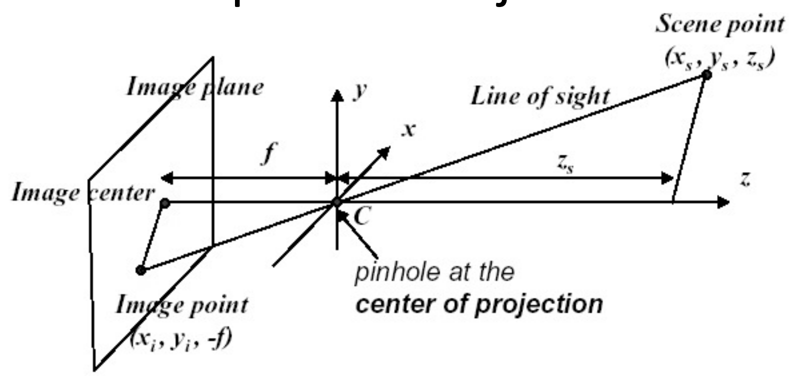


Perspective Projection



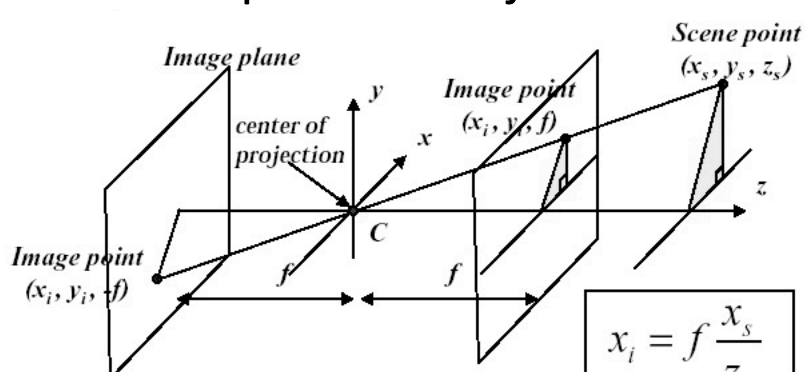
- Coordinate system center at the pinhole (center of projection).
- Image plane parallel to xy plane at distance f (focal length)
- Image center: intersection of z axis with image plane

Perspective Projection



- The point on the image plane that corresponds to a particular point in the scene is found by following the line that passes through the scene point and the center of projection.
- *Line of sight* to a point in the scene is the line through the center of projection to that point

Perspective Projection

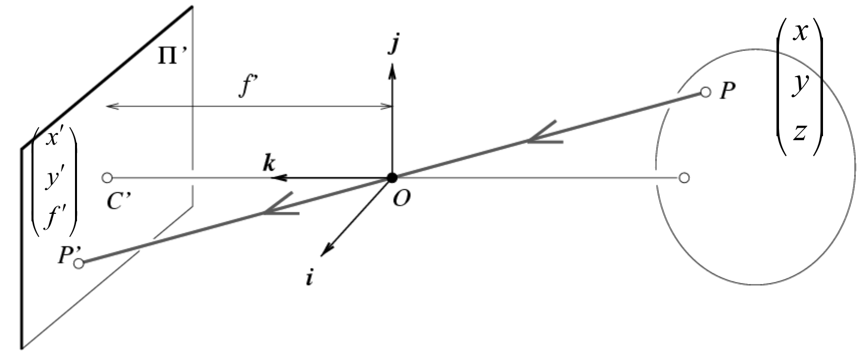


Fundamental equations for perspective projection onto a plane

$$x_i = f\,\frac{x_s}{z_s}$$

$$y_i = f\,\frac{y_s}{z_s}$$

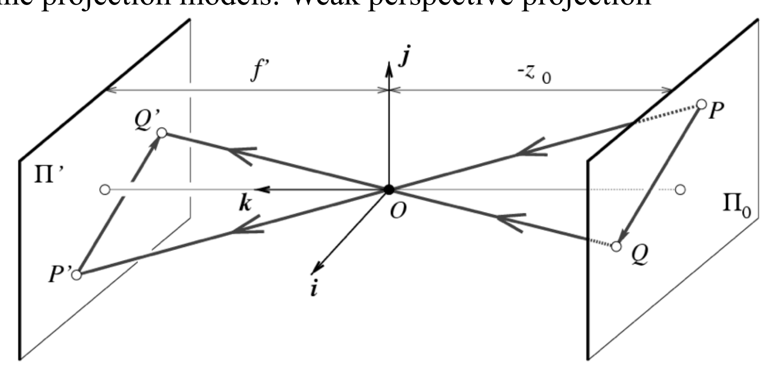
Pinhole Perspective Equation



$$\begin{cases} x' = f'\dfrac{x}{z} \\ y' = f'\dfrac{y}{z} \end{cases}$$

NOTE: *z* is always negative..

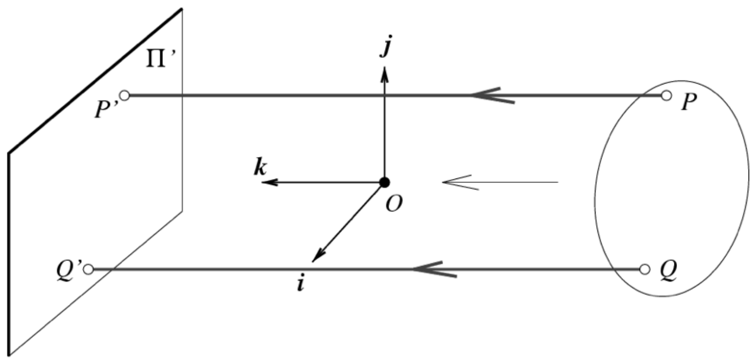
Affine projection models: Weak perspective projection



$$\begin{cases} x' = f'\dfrac{x}{z} \\ y' = f'\dfrac{y}{z} \end{cases} \Rightarrow \boxed{\begin{cases} x' = -mx \\ y' = -my \end{cases}}$$ where $m = -\dfrac{f'}{z_0}$ is the magnification.

- When the scene depth is small compared its distance from the Camera, we can assume every thing is on one plane,
- m can be taken constant: weak perspective projection
- also called scaled orthography (every thing is a scaled version of the scene)

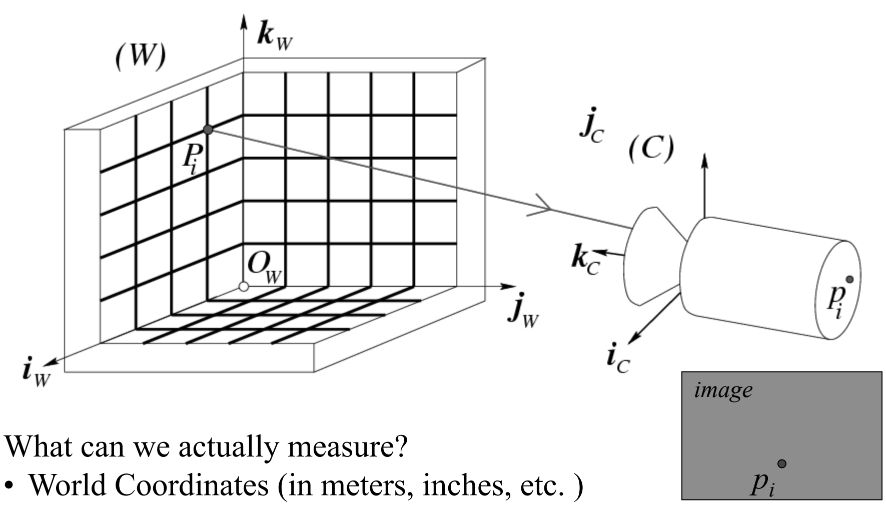
Affine projection models
Orthographic projection



$$\begin{cases} x' = x \\ y' = y \end{cases}$$

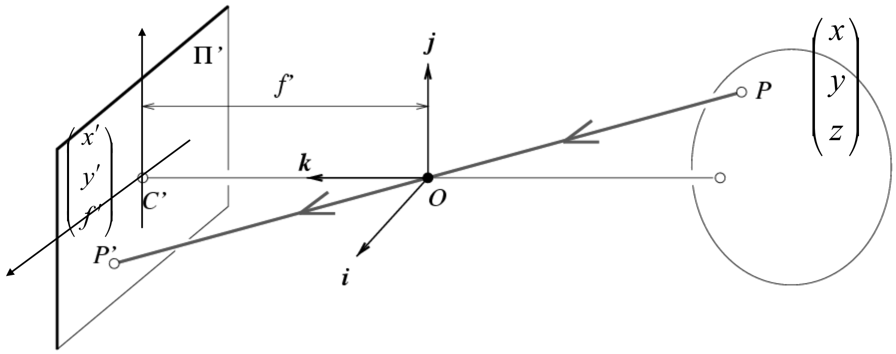
When the camera is at a (roughly constant) distance from the scene, take *m*=1.
All rays are parallel to *k* axis.

Quantitative Measurements and Calibration



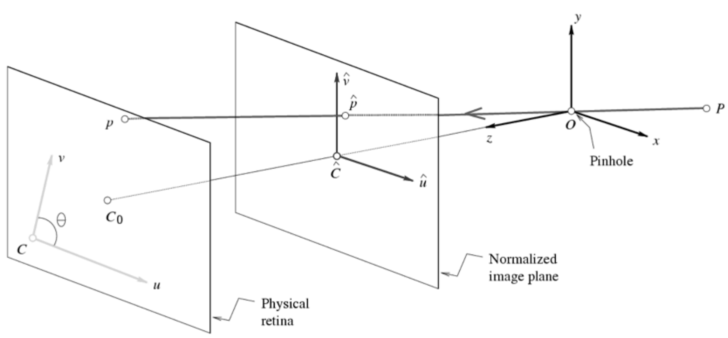
What can we actually measure?
- World Coordinates (in meters, inches, etc.)
- Image Coordinates (in pixels)
- How to relate these measurements ?

Pinhole Perspective Equation



$$\begin{cases} x' = f\,\dfrac{x}{z} \\ y' = f\,\dfrac{y}{z} \end{cases}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \frac{1}{z}\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



- We don't know where is the image center
- Pixels are rectangular
- Image axes are not necessary perpendicular (skew)

- Pixels are rectangular with scale parameters *k,l*

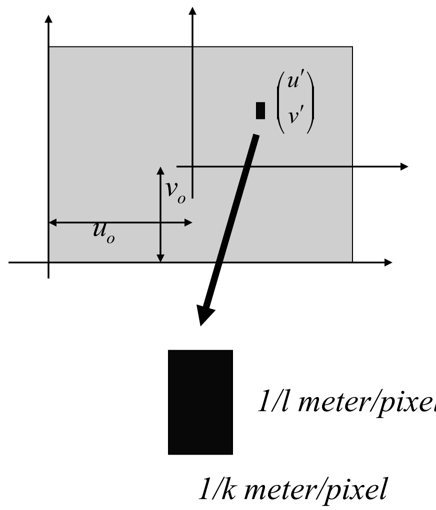
$$u' = kf\frac{x}{z} = \alpha\frac{x}{z}$$

$$v' = lf\frac{y}{z} = \beta\frac{y}{z}$$

- Move coordinate system to the corner

$$u = \alpha\frac{x}{z} + u_o$$

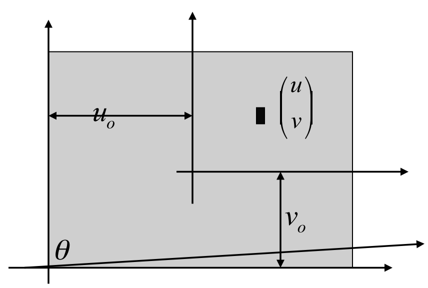
$$v = \beta\frac{y}{z} + v_o$$



- Pixel grid may not be exactly orthogonal
- $\theta \cong 90$ but not exactly

$$u = \alpha\frac{x}{z} + u_o$$

$$v = \beta\frac{y}{z} + v_o$$



$$u = \alpha\frac{x}{z} - \alpha\cot\theta\frac{y}{z} + u_o$$

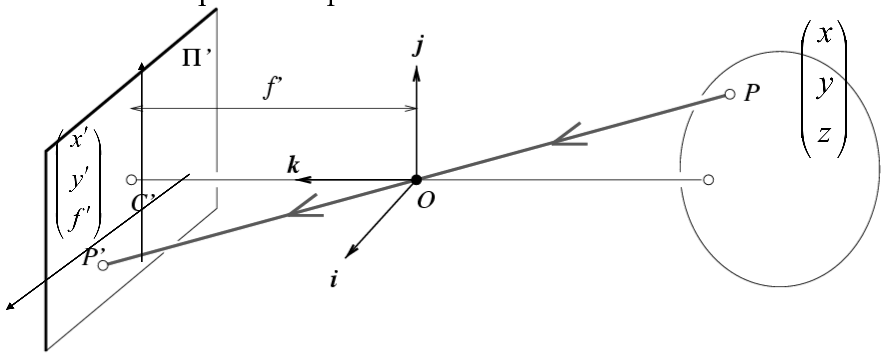
$$v = \frac{\beta}{\sin\theta}\frac{y}{z} + v_o$$

Approximation
s is skew parameter

$$u = \alpha\frac{x}{z} + s\frac{y}{z} + u_o$$

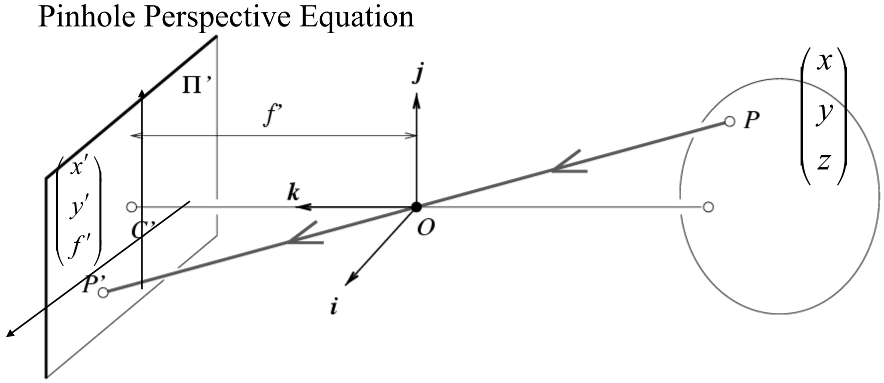
$$v = \beta\frac{y}{z} + v_o$$

Pinhole Perspective Equation



$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \frac{1}{z}\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \qquad \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \frac{1}{z}\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Pinhole Perspective Equation



$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \frac{1}{z}\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \qquad \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \frac{1}{z}\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \frac{1}{z}\begin{bmatrix} kf & 0 & u_0 & 0 \\ 0 & lf & v_o & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \frac{1}{z}\begin{bmatrix} \alpha & 0 & u_o & 0 \\ 0 & \beta & v_o & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \frac{1}{z}\begin{bmatrix} \alpha & -\alpha\cot\theta & u_o & 0 \\ 0 & \frac{\beta}{\sin\theta} & v_o & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \quad \text{OR} \quad \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \frac{1}{z}\begin{bmatrix} \alpha & s & u_o & 0 \\ 0 & \beta & v_o & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \frac{1}{z}\begin{bmatrix} \alpha & -\alpha\cot\theta & u_o & 0 \\ 0 & \frac{\beta}{\sin\theta} & v_o & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \quad \text{OR} \quad \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \frac{1}{z}\begin{bmatrix} \alpha & s & u_o & 0 \\ 0 & \beta & v_o & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- Five intrinsic camera parameters:
  - Magnification α, β *(in pixels)*
  - Image center location u_o, v_o *(in pixels)*
  - Skew measured as θ or s

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \frac{1}{z} \begin{bmatrix} \boxed{\begin{matrix} \alpha & -\alpha\cot\theta & u_o \\ 0 & \frac{\beta}{\sin\theta} & v_o \\ 0 & 0 & 1 \end{matrix}} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \quad \text{OR} \quad \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \frac{1}{z} \begin{bmatrix} \boxed{\begin{matrix} \alpha & s & u_o \\ 0 & \beta & v_o \\ 0 & 0 & 1 \end{matrix}} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\boldsymbol{p} = \frac{1}{z}\mathcal{M}\boldsymbol{P}, \quad \text{where} \quad \mathcal{M} \stackrel{\text{def}}{=} (\mathcal{K} \quad \mathbf{0})$$

- Five intrinsic camera parameters:
  - Magnification α, β *(in pixels)*
  - Image center location u_o, v_o *(in pixels)*
  - Skew measured as θ or *s*

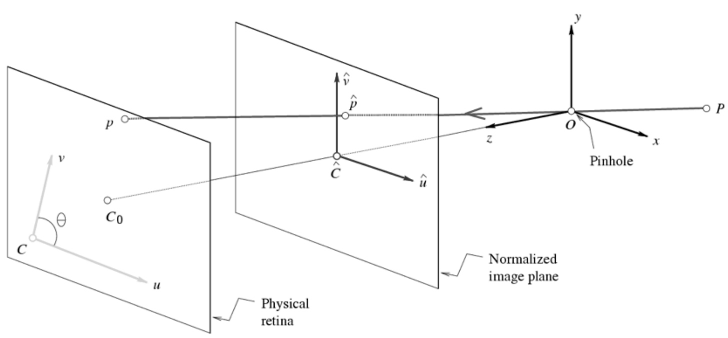
The Intrinsic Parameters of a Camera

Units:

k,l : pixel/m

f : m

α,β : pixel



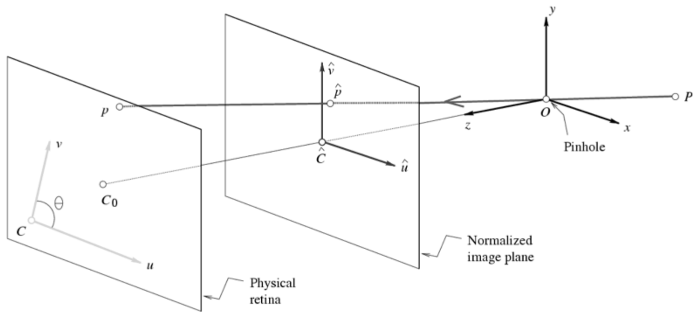
$$\begin{cases} \hat{u} = \dfrac{x}{z} \\ \hat{v} = \dfrac{y}{z} \end{cases} \Longleftrightarrow \hat{\boldsymbol{p}} = \frac{1}{z}(\mathrm{Id} \quad \mathbf{0})\begin{pmatrix} \boldsymbol{P} \\ 1 \end{pmatrix}$$

Physical Image Coordinates

Normalized Image Coordinates

$$\begin{cases} u = kf\dfrac{x}{z} \\ v = lf\dfrac{y}{z} \end{cases} \rightarrow \begin{cases} u = \alpha\dfrac{x}{z} + u_0 \\ v = \beta\dfrac{y}{z} + v_0 \end{cases} \rightarrow \begin{cases} u = \alpha\dfrac{x}{z} - \alpha\cot\theta\dfrac{y}{z} + u_0 \\ v = \dfrac{\beta}{\sin\theta}\dfrac{y}{z} + v_0 \end{cases}$$

The Intrinsic Parameters of a Camera



Calibration Matrix

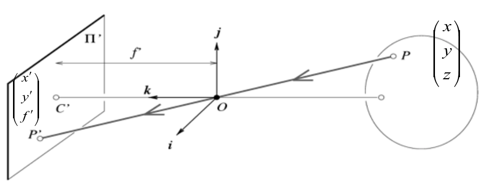
$$\boldsymbol{p} = \mathcal{K}\hat{\boldsymbol{p}}, \quad \text{where} \quad \boldsymbol{p} = \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} \quad \text{and} \quad \mathcal{K} \stackrel{\text{def}}{=} \begin{pmatrix} \alpha & -\alpha \cot\theta & u_0 \\ 0 & \dfrac{\beta}{\sin\theta} & v_0 \\ 0 & 0 & 1 \end{pmatrix}$$

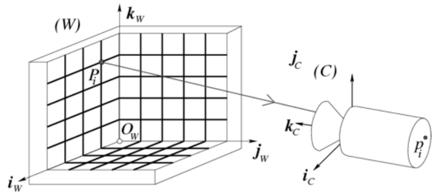
The Perspective Projection Equation

$$\boldsymbol{p} = \frac{1}{z}\mathcal{M}\boldsymbol{P}, \quad \text{where} \quad \mathcal{M} \stackrel{\text{def}}{=} \begin{pmatrix} \mathcal{K} & \boldsymbol{0} \end{pmatrix}$$

Extrinsic Parameters:

- Everything in the world so far is measured as if the pinhole is the coordinate center.
- Let's move to a real world coordinate system
- Where is the pinhole in the world coordinate system? [translation – 3 parameters]
- What is the orientation of the camera ? [rotation – 3 parameters]





$$\begin{pmatrix} {}^{C}P \\ 1 \end{pmatrix} = \begin{pmatrix} {}^{C}_{W}R_{3\times3} & {}^{C}O_{W\,3\times1} \\ 0^{T}{}_{1\times3} & 1 \end{pmatrix}_{4\times4} \begin{pmatrix} {}^{W}P \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} {}^{C}P \\ 1 \end{pmatrix} = \begin{pmatrix} {}^{C}_{W}R_{3\times3} & {}^{C}O_{W\,3\times1} \\ 0^{T}{}_{1\times3} & 1 \end{pmatrix}_{4\times4} \begin{pmatrix} {}^{W}P \\ 1 \end{pmatrix}$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \boxed{\begin{matrix} \alpha & -\alpha\cot\theta & u_o \\ 0 & \frac{\beta}{\sin\theta} & v_o \\ 0 & 0 & 1 \end{matrix}} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} K_{3\times3} & 0_{3\times1} \end{bmatrix} \begin{pmatrix} {}^{C}_{W}R_{3\times3} & {}^{C}O_{W\,3\times1} \\ 0^{T}{}_{1\times3} & 1 \end{pmatrix}_{4\times4} \begin{pmatrix} {}^{W}P \\ 1 \end{pmatrix}$$

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} K_{3\times3} & 0_{3\times1} \end{bmatrix} \begin{pmatrix} {}^{C}_{W}R_{3\times3} & {}^{C}O_{W\,3\times1} \\ 0^{T}{}_{1\times3} & 1 \end{pmatrix}_{4\times4} \begin{pmatrix} {}^{W}P \\ 1 \end{pmatrix}$$

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = K_{3\times3} \begin{pmatrix} {}^{C}_{W}R_{3\times3} & {}^{C}O_{W\,3\times1} \end{pmatrix}_{3\times4} \begin{pmatrix} {}^{W}P \\ 1 \end{pmatrix} \Rightarrow p = MP$$

Rotation *Translation*

Intrinsic parameters

Extrinsic parameters

Projection matrix

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = K_{3\times3}\begin{pmatrix} {}^C_W R_{3\times3} & {}^C O_{W\,3\times1} \end{pmatrix}_{3\times4} \begin{pmatrix} {}^W P \\ 1 \end{pmatrix} \Rightarrow p = MP$$

Rotation *Translation*

Intrinsic parameters

Extrinsic parameters

Projection matrix
3x4

$$M = K_{3\times3}\begin{pmatrix} {}^C_W R_{3\times3} & {}^C O_{W\,3\times1} \end{pmatrix}_{3\times4}$$

Only 11 free parameters (not 12):
5 intrinsic, 3 for rotation, 3 for translation
Perspective Projection Matrix

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = K_{3\times3}\begin{pmatrix} {}^C_W R_{3\times3} & {}^C O_{W\,3\times1} \end{pmatrix}_{3\times4} \begin{pmatrix} {}^W P \\ 1 \end{pmatrix} \Rightarrow p = MP$$

$$M = K_{3\times3}\begin{pmatrix} {}^C_W R_{3\times3} & {}^C O_{W\,3\times1} \end{pmatrix}_{3\times4}$$

Replacing $\mathcal{M}$ by $\lambda\mathcal{M}$ in

$$\begin{cases} u = \dfrac{\boldsymbol{m}_1 \cdot \boldsymbol{P}}{\boldsymbol{m}_3 \cdot \boldsymbol{P}} \\ v = \dfrac{\boldsymbol{m}_2 \cdot \boldsymbol{P}}{\boldsymbol{m}_3 \cdot \boldsymbol{P}} \end{cases}$$

does not change u and v.

M is only defined up to scale in this setting!!

Explicit Form of the Projection Matrix

$$\mathcal{M} = \begin{pmatrix} \boxed{\begin{matrix} \alpha \boldsymbol{r}_1^T - \alpha \cot\theta \boldsymbol{r}_2^T + u_0 \boldsymbol{r}_3^T \\ \dfrac{\beta}{\sin\theta}\boldsymbol{r}_2^T + v_0 \boldsymbol{r}_3^T \\ \boldsymbol{r}_3^T \end{matrix}} & \boxed{\begin{matrix} \alpha t_x - \alpha \cot\theta t_y + u_0 t_z \\ \dfrac{\beta}{\sin\theta} t_y + v_0 t_z \\ t_z \end{matrix}} \end{pmatrix}$$

Note: If $\mathcal{M} = (\mathcal{A} \quad \boldsymbol{b})$ then $|\boldsymbol{a}_3| = 1$.

$$p = MP = \begin{pmatrix} m_1^T \\ m_2^T \\ m_3^T \end{pmatrix} P \quad \Longrightarrow \quad u = \frac{\boldsymbol{m}_1^T \cdot \boldsymbol{P}}{\boldsymbol{m}_3^T \cdot \boldsymbol{P}} \qquad v = \frac{\boldsymbol{m}_2^T \cdot \boldsymbol{P}}{\boldsymbol{m}_3^T \cdot \boldsymbol{P}}$$

Replacing *M* by λM doesn't change *u* or *v*

M is only defined up to scale in this setting!!

Theorem (Faugeras, 1993)

Let $\mathcal{M} = (\mathcal{A} \quad \boldsymbol{b})$ be a 3×4 matrix and let $\boldsymbol{a}_i^T$ $(i = 1, 2, 3)$ denote the rows of the matrix $\mathcal{A}$ formed by the three leftmost columns of $\mathcal{M}$.

- A necessary and sufficient condition for $\mathcal{M}$ to be a perspective projection matrix is that $\text{Det}(\mathcal{A}) \neq 0$.
- A necessary and sufficient condition for $\mathcal{M}$ to be a zero-skew perspective projection matrix is that $\text{Det}(\mathcal{A}) \neq 0$ and

$$(\boldsymbol{a}_1 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3) = 0.$$

- A necessary and sufficient condition for $\mathcal{M}$ to be a perspective projection matrix with zero skew and unit aspect-ratio is that $\text{Det}(\mathcal{A}) \neq 0$ and

$$\begin{cases} (\boldsymbol{a}_1 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3) = 0, \\ (\boldsymbol{a}_1 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_1 \times \boldsymbol{a}_3) = (\boldsymbol{a}_2 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3). \end{cases}$$

Camera Calibration

- Find the intrinsic and extrinsic parameters of a camera
- VERY large literature on the subject
- Work of Roger Tsai influential
- Basic idea: Given a set of world points P_i and their image coordinates (u_i,v_i) find the projection matrix and then find intrinsic and extrinsic parameters.

Sources

- Forsyth and Ponce, Computer Vision a Modern approach: 1.1, 2.1,2.2
- Fougeras, Three-dimensional Computer Vision
- Slides by J. Ponce UIUC
- Slides by L. S. Davis UMD