CS443: Digital Imaging and Multimedia
Binary Image Analysis

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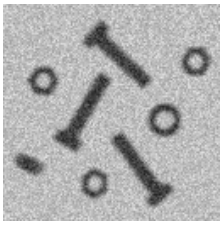
Outlines

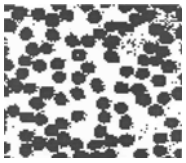
- A Simple Machine Vision System
- Image segmentation by thresholding
- Digital geometry
- Connected components
- Mathematical morphology
- Region descriptors
- Limitations
- Sources:
  - Burger and Burge “Digital Image Processing” Chapter 11
  - Forsyth and Ponce “Computer Vision a Modern approach”
  - Slides by Prof. L. Davis @ UMD

Problem definition

Objective: Design a vision system to “see” a “flat” world

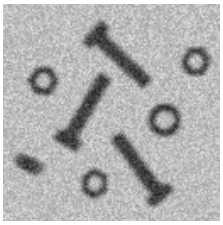
- Page of text: text, line drawings, etc.
- Side panel of a truck
- Objects on an inspection belt
- X-ray image of separated potatoes
- Microscopic image of blood cells
- etc.

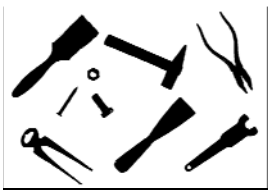




Problem definition

- General approach to recognition/inspection
  - Acquire gray scale image using camera
  - Reduce to black and white image - black objects on white background
  - Find individual black objects and measure their properties
  - Compare those properties to object models





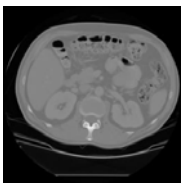


Image segmentation

- How do we know which groups of pixels in a digital image correspond to the objects to be analyzed?
  - objects may be uniformly darker or brighter than the background against which they appear
    - black characters imaged against the white background of a page
    - bright, dense potatoes imaged against a background that is transparent to X-rays

Viewed as a kind of invention by evolution, the cerebral cortex must be one of the great success stories in the history of living things. In vertebrates lower than mammals the cerebral cortex is minuscule, if it can be said to exist at all. Suddenly impressive in the lowest mammals, it begins to dominate the brain in carnivores, and it increases explosively in primates; in man it almost completely envelops the

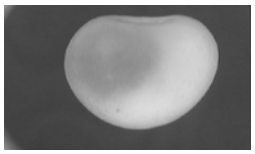
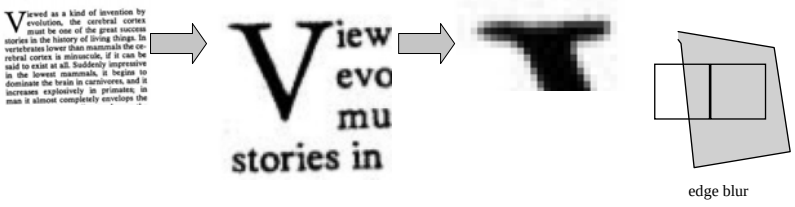


Image segmentation

- Ideally, object pixels would be black (0 intensity) and background pixels white (maximum intensity)
- But this rarely happens
  - pixels overlap regions from both the object and the background, yielding intensities between pure black and white - edge blur
  - cameras introduce “noise” during imaging - measurement “noise”
  - e.g., potatoes have non-uniform “thickness”, giving variations in brightness in X-ray - model “noise”

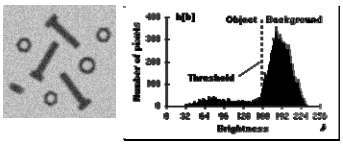


edge blur

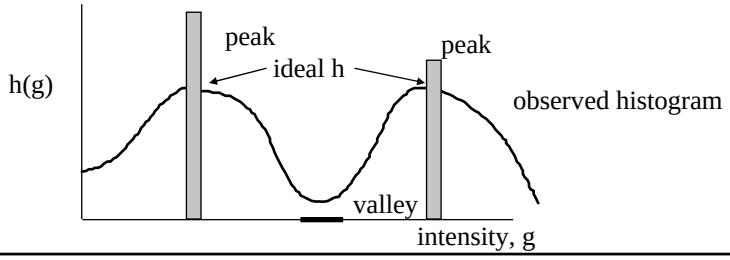
Image segmentation by thresholding

- But if the objects and background occupy different ranges of gray levels, we can "mark" the object pixels by a process called **thresholding:**
  - Let I(u,v) be the original, gray level image
  - B(u,v) is a binary image (pixels are either 0 or 1) created by thresholding I(u,v)
    - B(u,v) = 1 if I(u,v) < t
    - B(u,v) = 0 if I(u,v) >= t
    - We will assume that the 1's are the object pixels and the 0's are the background pixels
- Thresholding is a point operation
- How to choose the threshold t ?

Thresholding



- How do we choose the threshold t?
- Histogram (h) - gray level frequency distribution of the gray level image I.
  - $h_I(g)$ = number of pixels in F whose gray level is g
  - $H_I(g)$ = number of pixels in F whose gray level is <=g (cumulative)

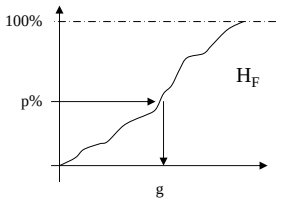


Thresholding

- Peak and valley method
  - Find the two most prominent peaks of h
    - g is a peak if $h(g) > h(g \pm \Delta g)$, $\Delta g = 1, ..., k$
  - Let g_1 and g_2 be the two highest peaks, with $g_1 < g_2$
  - Find the deepest valley, g, between g_1 and g_2
    - g is the valley if $h_F(g) <= h_F(g')$, g,g' in [g1, g2]
  - Use g as the threshold
  - Problem: histograms are not smooth. How can we smooth a histogram?

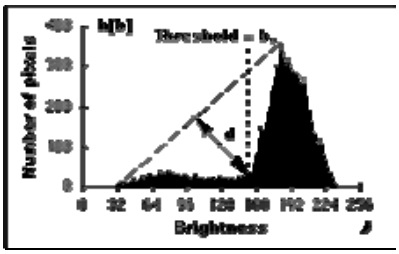
Thresholding

- P-tile method
  - in some applications we know approximately what percentage, p, of the pixels in the image come from objects
    - might have one potato in the image, or one character.
  - Cumulative histogram $H_I(g)$ can be used to find the gray level, g, such that ~p% of the pixels have intensity <= g
  - Then, we can examine h_I in the neighborhood of g to find a good threshold (low valley point)



Triangle algorithm

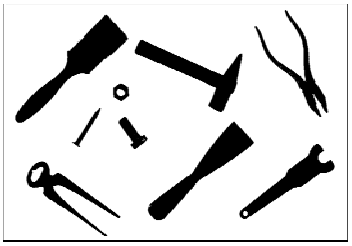
- A line is constructed between the maximum of the histogram at brightness b_{max} and the lowest value b_{min} = (p=0)% in the image.
- The distance d between the line and the histogram h[b] is computed for all values of b from $b = b_{min}$ to $b = b_{max}$.
- The brightness value b_o where the distance between $h[b_o]$ and the line is maximal is the threshold value.
- This technique is particularly effective when the object pixels produce a weak peak in the histogram.

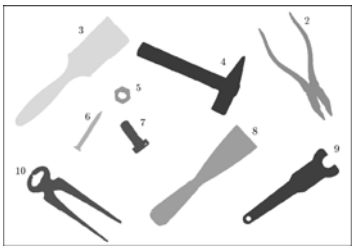


Thresholding

- Hand selection
  - Arbitrary select a threshold by hand!
- Many threshold selection methods in the literature
  - Probabilistic methods
    - make parametric assumptions about object and background intensity distributions and then derive “optimal” thresholds
  - Structural methods
    - Evaluate a range of thresholds wrt properties of resulting binary images
      - one with straightest edges, maximum contrast, most easily recognized objects, etc.
  - Local thresholding
    - apply thresholding methods to image windows

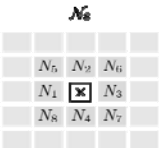
- What's next after thresholding
- We need to detect separate regions in the binary image.
- This process is called:
  - Region labeling
  - Connected component labeling





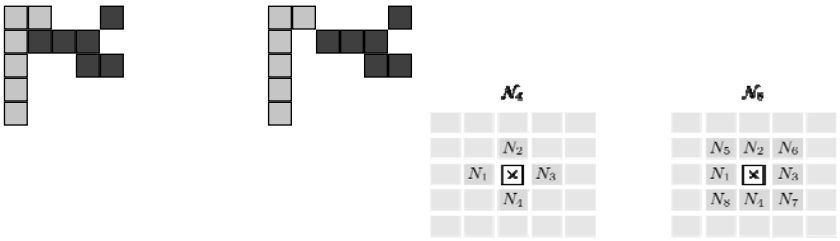
Neighborhood and Connectivity

- Definition: Given a pixel (i,j) its 4-neighbors are the points (i',j') such that |i-i'| + |j-j'| = 1
  - the 4-neighbors are (i±1, j) and (i,j±1)
- Definition: Given a pixel (i,j) its 8-neighbors are the points (i',j') such that max(|i-i'|,|j-j'|) = 1
  - the 8- neighbors are (i, j±1), (i±1, j) and (i±1, j±1)

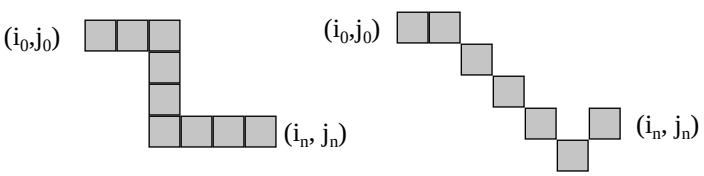


Connectivity

- 4-connectivity vs. 8-connectivity:
- Definition: Given two disjoint sets of pixels, A and B, A is 4(8)-connected to B if there is a pixel in A that is a 4(8) neighbor of a pixel in B
- So, we have to chose 4 or 8 neighborhood notion to use.Adjacency
- Definition: Given two disjoint sets of pixels, A and B, A is 4-(8) adjacent to B: there is a pixel in A that is a 4-(8) neighbor of a pixel in B



- Definition: A 4-(8)path from pixel (i_0,j_0) to (i_n,j_n) is a sequence of pixels (i_0,j_0) (i_1,j_1) (i_2,j_2) , ... (i_n,j_n) such that (i_k, j_k) is a 4-(8) neighbor of (i_{k+1}, j_{k+1}), for k = 0, ..., n-1



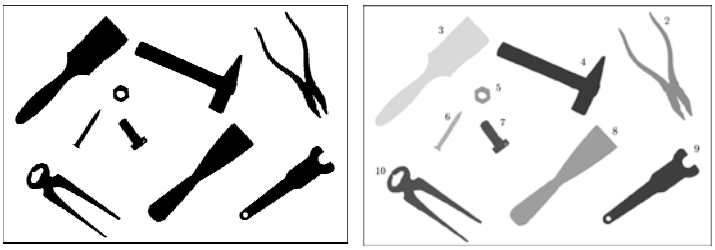
Every 4-path is an 8-path!

Connected-to relation

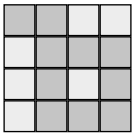
- Definition: Given a binary image, B, the set of all 1's is called the foreground and is denoted by S
- Definition: Given a pixel p in S, p is 4(8) connected to q in S if there is a path from p to q consisting only of points from S using 4(8) neighbors.
- The relation "is-connected-to" is an equivalence relation
  - Reflexive - p is connected to itself by a path of length 0
  - Symmetric - if p is connected to q, then q is connected to p by the reverse path
  - Transitive - if p is connected to q and q is connected to r, then p is connected to r by concatenation of the paths from p to q and q to r

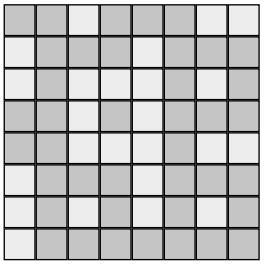
Connected components

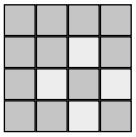
- Since the "is-connected-to" relation is an equivalence relation, it partitions the set S into a set of equivalence classes or components
  - these are called connected components, or connected regions



Examples - Yellow = 1, Green = 0



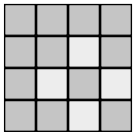




How many 4- (8) components of S?
What is the background?
Which are the 4- (8) holes?

Background and foreground connectivity

- Use opposite connectivity for the foreground and the background
  - 4-foreground, 8-background: 4 single pixel objects and no holes
  - 4-background, 8-foreground: one 4 pixel object containing a 1 pixel hole



FloodFill region labeling

```
1:  RegionLabeling(I)
        I: binary image (0 = background, 1 = foreground)
        The image I is labeled (destructively modified) and returned.
2:      Initialize m ← 2 (the value of the next label to be assigned).
3:      Iterate over all image coordinates (u, v).
4:          If I(u, v) = 1 then
5:              FloodFill(I, u, v, m)        ▷ use any of the 3 versions below
6:              m ← m + 1.
7:      return the labeled image I.

8:  FloodFill(I, u, v, label)                ▷ Recursive Version
9:      if coordinate (u, v) is within image boundaries and I(u, v) = 1 then
10:         Set I(u, v) ← label
11:         FloodFill(I, u+1, v, label)
12:         FloodFill(I, u, v+1, label)
13:         FloodFill(I, u, v−1, label)
14:         FloodFill(I, u−1, v, label)
15:     return.

16: FloodFill(I, u, v, label)                ▷ Depth-First Version
17:     Create an empty stack S
```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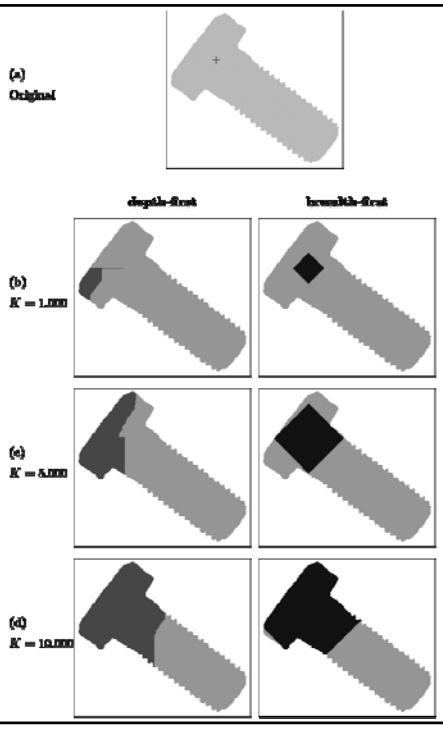
14:         FloodFill(I, u−1, v, label)
15:     return.
```


```
16: FloodFill(I, u, v, label)                ▷ Depth-First Version
17:     Create an empty stack S
18:     Put the seed coordinate (u, v) onto the stack: Push(S, (u, v))
19:     while S is not empty do
20:         Get the next coordinate from the top of the stack:
            (x, y) ← Pop(S)
21:         if coordinate (x, y) is within image boundaries and I(x, y) = 1
                then
22:             Set I(x, y) ← label
23:             Push(S, (x+1, y))
24:             Push(S, (x, y+1))
25:             Push(S, (x, y−1))
26:             Push(S, (x−1, y))
27:     return.

28: FloodFill(I, u, v, label)                ▷ Breadth-First Version
29:     Create an empty queue Q
30:     Insert the seed coordinate (u, v) into the queue: Enqueue(Q, (u, v))
31:     while Q is not empty do
32:         Get the next coordinate from the front of the queue:
            (x, y) ← Dequeue(Q)
33:         if coordinate (x, y) is within image boundaries and I(x, y) = 1
                then
34:             Set I(x, y) ← label
35:             Enqueue(Q, (x+1, y))
36:             Enqueue(Q, (x, y+1))
37:             Enqueue(Q, (x, y−1))
38:             Enqueue(Q, (x−1, y))
39:     return.
```



Sequential Region Labeling

- More efficient two-pass algorithm for region labeling

(a)

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	1	0	0	1	1	0	1	0	
0	1	1	1	1	1	1	0	0	1	0	0	1	0	
0	0	0	0	1	0	1	0	0	0	0	0	1	0	
0	1	1	1	1	1	1	1	1	1	1	1	1	0	
0	0	0	0	1	1	1	1	1	1	1	1	1	0	
0	1	1	0	0	0	1	0	1	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	

0 Background

1 Foreground

$N_4(u,v) =$

	N_2	
N_1	×	

or $N_8(u,v) =$

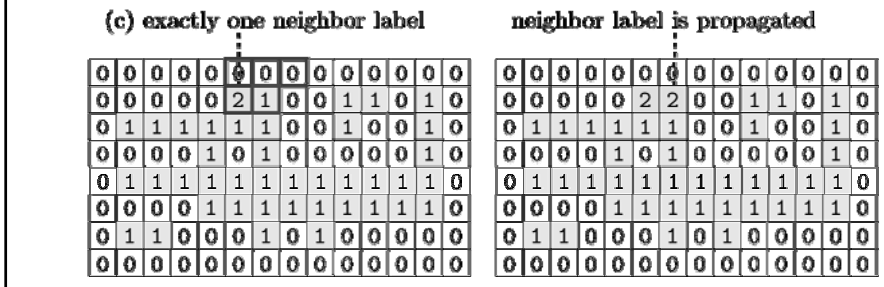
N_2	N_3	N_4
N_1	×	

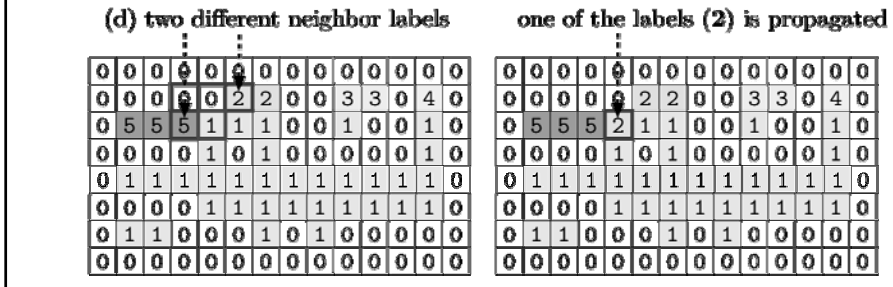
(b) only background neighbors

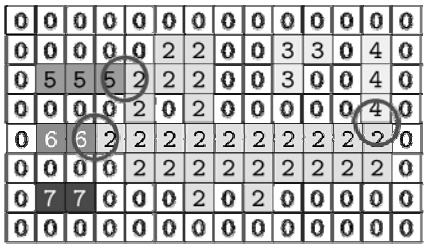
new label (2)

0	0	0	0	0	0	0	0	0	0	0	0	0	0	
0	0	0	0	0	1	1	0	0	1	1	0	1	0	
0	1	1	1	1	1	1	0	0	1	0	0	1	0	
0	0	0	0	1	0	1	0	0	0	0	0	1	0	
0	1	1	1	1	1	1	1	1	1	1	1	1	0	
0	0	0	0	1	1	1	1	1	1	1	1	1	0	
0	1	1	0	0	0	1	0	1	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	

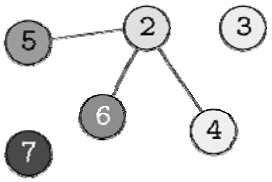
0	0	0	0	0	2	0	0	0	0	0	0	0	0	
0	0	0	0	0	2	1	0	0	1	1	0	1	0	
0	1	1	1	1	1	1	0	0	1	0	0	1	0	
0	0	0	0	1	0	1	0	0	0	0	0	1	0	
0	1	1	1	1	1	1	1	1	1	1	1	1	0	
0	0	0	0	1	1	1	1	1	1	1	1	1	0	
0	1	1	0	0	0	1	0	1	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	


(c) exactly one neighbor label
neighbor label is propagated

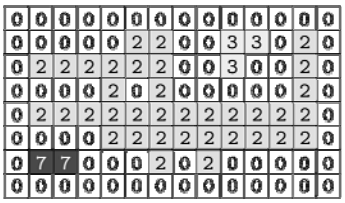

(d) two different neighbor labels
one of the labels (2) is propagated

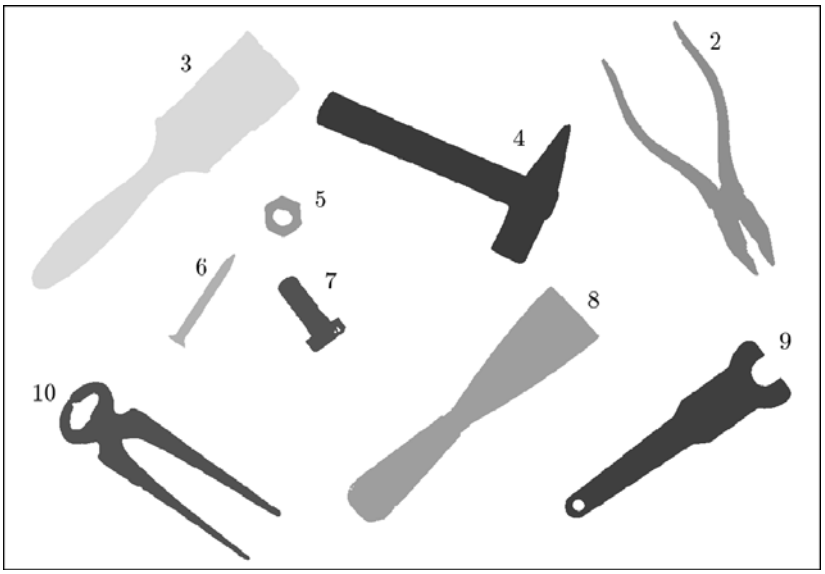


(a)

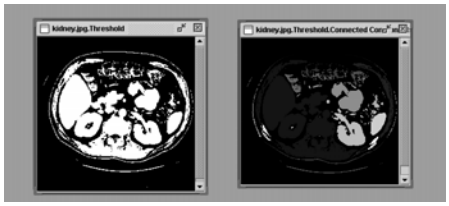


(b)





- Things might not be perfect!
- Different regions might be connected due to noise or wrong threshold.
- What to do about that?



Mathematical Morphology

Binary mathematical morphology consists of two basic operations

dilation and erosion

and several composite relations

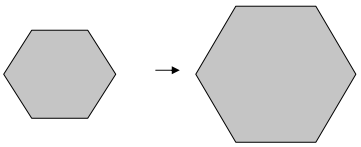
closing and opening
conditional dilation
. . .

Dilation

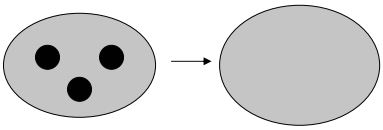
Dilation expands the connected sets of 1s of a binary image.

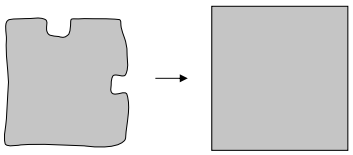
It can be used for

1. growing features



2. filling holes and gaps



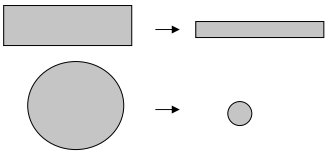


Erosion

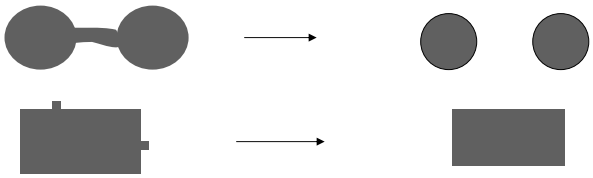
Erosion shrinks the connected sets of 1s of a binary image.

It can be used for

1. shrinking features



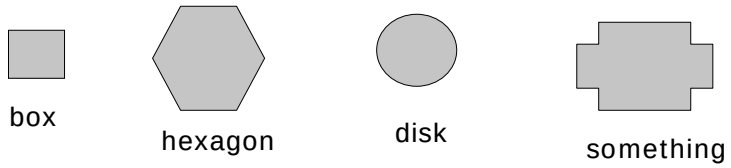
2. Removing bridges, branches and small protrusions



Structuring Elements

A structuring element is a shape mask used in the basic morphological operations.

They can be any shape and size that is digitally representable, and each has an origin.



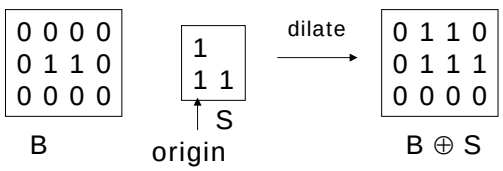
box(length,width) disk(diameter)

Dilation with Structuring Elements

The arguments to dilation and erosion are

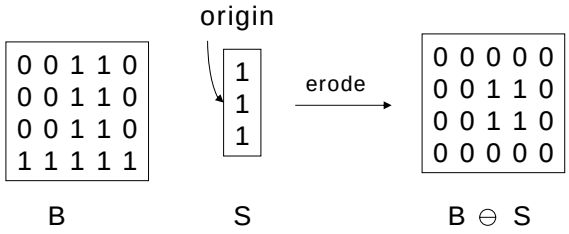
1. **a binary image B**
2. **a structuring element S**

dilate(B,S) takes binary image B, places the origin of structuring element S over each 1-pixel, and ORs the structuring element S into the output image at the corresponding position.

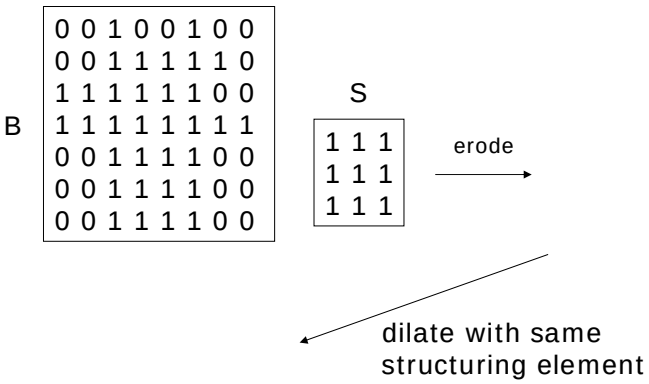


Erosion with Structuring Elements

erode(B,S) takes a binary image B, places the origin of structuring element S over every pixel position, and ORs a binary 1 into that position of the output image only if every position of S (with a 1) covers a 1 in B.



Example 1 to Try



Example 2 to Try

B

0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0
0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0
0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0
0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0
0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

S

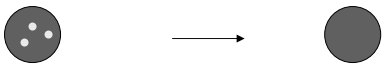
0	1	1	1	0
1	1	1	1	1
1	1	1	1	1
1	1	1	1	1
0	1	1	1	0

First erode and then dilate with the same S.

Opening and Closing

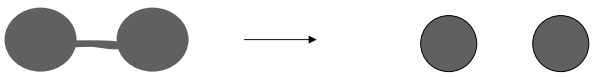
- Closing is the compound operation of dilation followed by erosion (with the same structuring element)

$$B \bullet S = (B \oplus S) - S$$



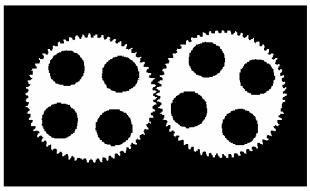
- Opening is the compound operation of erosion followed by dilation (with the same structuring element)

$$B \circ S = (B - S) \oplus S$$

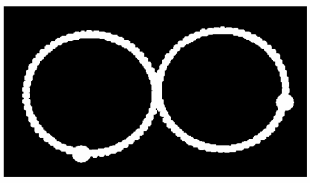


Gear Tooth Inspection

Sternberg 1985: use morphology to detect missing or broken watch gear teeth



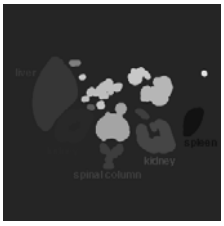
Original binary image



Detected defects

How did they do it?

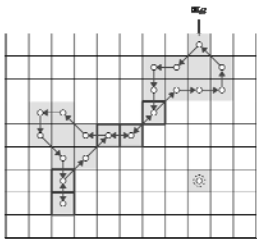
Connected Components after thresholding



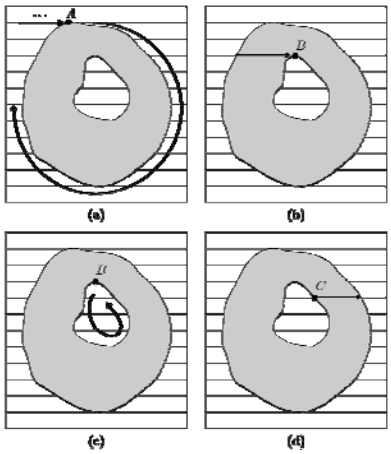
Connected Components after Morphology

Region Contours

- Tracing a region contour is very useful to extract a descriptor of the region.
- Difficulties:
  - External and internal contours
  - The contour might run through the same pixel more than once

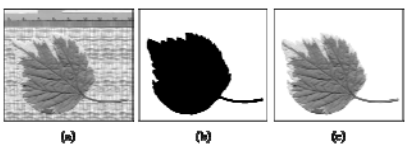


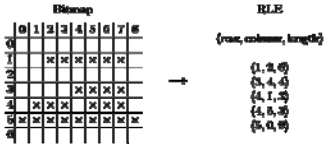
- Contour finding can be done after region labeling
- Algorithms for both region labeling and contour finding: when you first hit a contour, trace it.
- See textbook 11.2.2 and 11.2.3



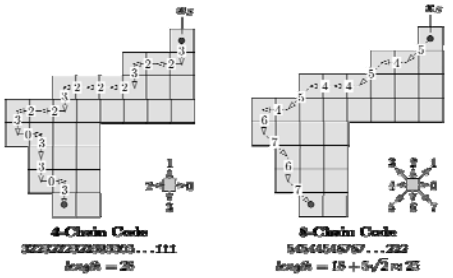
Representing image regions

- Matrix representation: 2D array, mask, "bitmap"
- Run Length encoding: provides a way for lossless compression.
  - foundation for fax transmission.
  - used in many image formats (among other methods) bmp, TIFF, GIF, JPEG



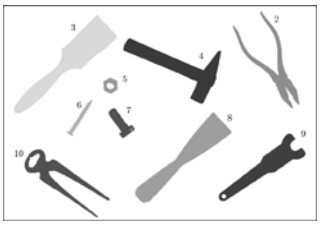


- Chain Codes: (Freeman codes) a method for contour encoding
- Begin at a given point and represent the contour by a sequence of discrete directional changes
  - Absolute chain code $c'_R = [5, 4, 5, 4, 4, 5, 4, 6, 7, 6, 7, \ldots 2, 2, 2]$
  - Differential chain code $c''_R = [7, 1, 7, 0, 1, 7, 2, 1, 7, 1, 1, \ldots 0, 0, 3]$



Properties of Binary Regions

- Our goal is to recognize each connected component as one of a set of known objects
  - letters of the alphabet
  - good potatoes versus bad potatoes
- We need to associate measurements, or properties, with each connected component that we can compare against expected properties of different object types.
- An important aspect is the invariance of the descriptor:
  - Invariant to translation
  - Invariant to scaling (size)
  - Invariant to rotation
  - Other invariants



Properties of Binary Regions

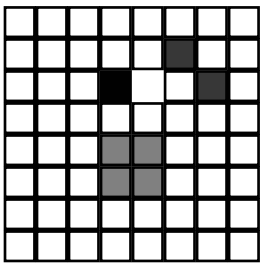
- Geometric Features
- Statistical Shape Properties
- Moment-based geometrical properties
- Projections
- ….

Geometric Features

- Perimeter
- Area
- Compactness and Roundness
- Bounding box
- Convex hull

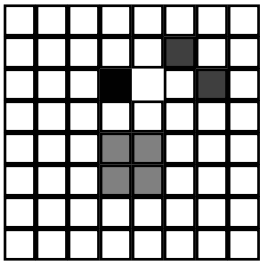
How do we compute the perimeter of a connected component?

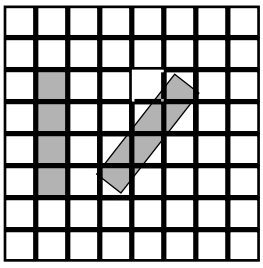
1. Count the number of pixels in the component adjacent to 0's
   - perimeter of black square would be 1
   - but perimeter of gray square, which has 4x the area, would be 4
   - but perimeter should go up as sqrt of area
2. Count the number of 0's adjacent to the component
   - works for the black and gray squares, but fails for the red dumbbell



How do we compute the perimeter of a connected component?

3) Count the number of sides of pixels in the component adjacent to 0's
   - these are the **cracks** between the pixels
   - clockwise traversal of these cracks is called a crack code
   - perimeter of black is 4, gray is 8 and red is 8
- What effect does rotation have on the value of a perimeter of the digitization of a simple shape?
  - rotation can lead to large changes in the perimeter and the area!



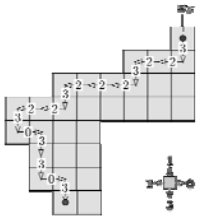


- Perimeter from a chain code

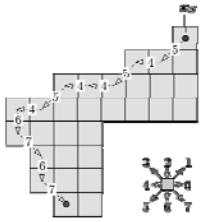
$$\text{Perimeter}(\mathcal{R}) = \sum_{i=0}^{M-1} \text{length}(c_i')$$

$$\text{with} \quad \text{length}(c) = \begin{cases} 1 & \text{for } c = 0, 2, 4, 6 \\ \sqrt{2} & \text{for } c = 1, 3, 5, 7 \end{cases}$$

$$P(\mathcal{R}) \approx \text{Perimeter}_{\text{corr}}(\mathcal{R}) = 0.95 \cdot \text{Perimeter}(\mathcal{R})$$



4-Chain Code



8-Chain Code

- Perimeter

$$\text{Perimeter}(\mathcal{R}) = \sum_{i=0}^{M-1} \text{length}(c_i')$$

$$\text{with} \quad \text{length}(c) = \begin{cases} 1 & \text{for } c = 0, 2, 4, 6 \\ \sqrt{2} & \text{for } c = 1, 3, 5, 7 \end{cases}$$

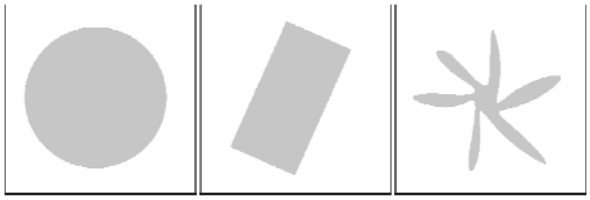
- Area

$$A(\mathcal{R}) = |\mathcal{R}| = N.$$

- Compactness and Roundness

$$\text{Circularity}(\mathcal{R}) = 4\pi \cdot \frac{A(\mathcal{R})}{P^2(\mathcal{R})}$$

$$\text{Circularity}(\mathcal{R}) \approx 4\pi \cdot \frac{A(\mathcal{R})}{\text{Perimeter}^2_{\text{corr}}(\mathcal{R})}$$

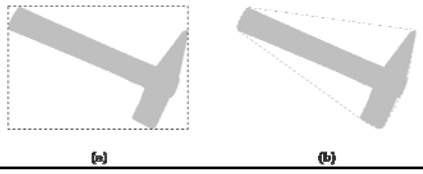


1.001 0.672 0.086

- Bounding box: minimal axis-parallel rectangle that encloses all the points of the region

$$\text{BoundingBox}(\mathcal{R}) = \langle u_{\min}, u_{\max}, v_{\min}, v_{\max} \rangle$$

- Convex hull: the smallest polygon within which the all points in the region fit.
  - Convexity: relationship between the length of the convex hull and the perimeter of the region
  - Density: the ratio between the area of the region and the area of the convex hull



(a) (b)

Statistical shape properties

- Centroid $\bar{x} = \frac{1}{|\mathcal{R}|} \cdot \sum_{(u,v)\in\mathcal{R}} u$ and $\bar{y} = \frac{1}{|\mathcal{R}|} \cdot \sum_{(u,v)\in\mathcal{R}} v$
- Moments

$$m_{pq} = \sum_{(u,v)\in\mathcal{R}} I(u,v) \cdot u^p v^q$$

For binary images: $m_{pq} = \sum_{(u,v)\in\mathcal{R}} u^p v^q$

$$A(\mathcal{R}) = |\mathcal{R}| = \sum_{(u,v)\in\mathcal{R}} 1 = \sum_{(u,v)\in\mathcal{R}} u^0 v^0 = m_{00}(\mathcal{R})$$

$$\bar{x} = \frac{1}{|\mathcal{R}|} \cdot \sum_{(u,v)\in\mathcal{R}} u^1 v^0 = \frac{m_{10}(\mathcal{R})}{m_{00}(\mathcal{R})} \qquad \bar{y} = \frac{1}{|\mathcal{R}|} \cdot \sum_{(u,v)\in\mathcal{R}} u^0 v^1 = \frac{m_{01}(\mathcal{R})}{m_{00}(\mathcal{R})}$$

- Central Moments: position-independent (translation invariant) moments: shift the coordinate system to the centroid:

$$\mu_{pq}(\mathcal{R}) = \sum_{(u,v)\in\mathcal{R}} (u-\bar{x})^p \cdot (v-\bar{y})^q$$

- Normalized central moments: invariant to the size (scale) of the region

$$\bar{\mu}_{pq}(\mathcal{R}) = \mu_{pq} \cdot \left(\frac{1}{\mu_{00}(\mathcal{R})}\right)^{(p+q+2)/2}$$

Geometric properties from moments

Orientation:

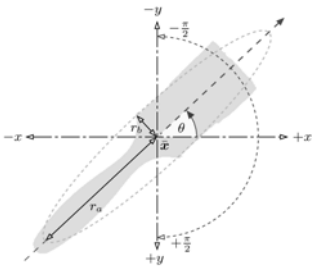
Direction of the major axis

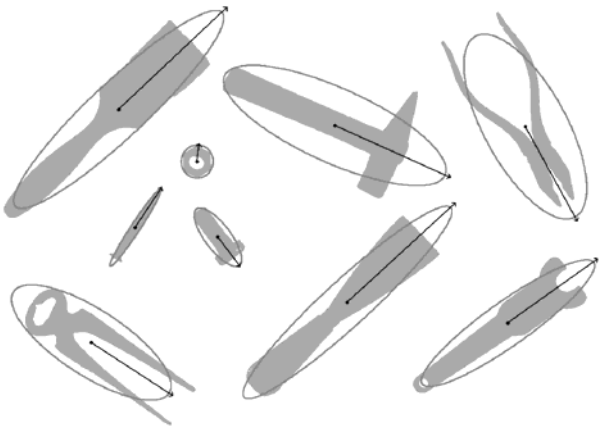
$$\tan(2\,\theta_{\mathcal{R}}) = \frac{2\cdot\mu_{11}(\mathcal{R})}{\mu_{20}(\mathcal{R}) - \mu_{02}(\mathcal{R})}$$

$$\theta_{\mathcal{R}} = \frac{1}{2}\tan^{-1}\left(\frac{2\cdot\mu_{11}(\mathcal{R})}{\mu_{20}(\mathcal{R}) - \mu_{02}(\mathcal{R})}\right)$$

Eccentricity: “elongatedness”

$$\mathrm{Ecc}(\mathcal{R}) = \frac{a_1}{a_2} = \frac{\mu_{20}+\mu_{02}+\sqrt{(\mu_{20}-\mu_{02})^2+4\cdot\mu_{11}^2}}{\mu_{20}+\mu_{02}-\sqrt{(\mu_{20}-\mu_{02})^2+4\cdot\mu_{11}^2}}$$





$$A = \begin{pmatrix} \mu_{20} & \mu_{11} \\ \mu_{11} & \mu_{02} \end{pmatrix} \qquad \mathrm{Ecc}(\mathcal{R}) = \frac{a_1}{a_2} = \frac{\mu_{20} + \mu_{02} + \sqrt{(\mu_{20} - \mu_{02})^2 + 4 \cdot \mu_{11}^2}}{\mu_{20} + \mu_{02} - \sqrt{(\mu_{20} - \mu_{02})^2 + 4 \cdot \mu_{11}^2}}$$

$$r_a = 2 \cdot \left(\frac{\lambda_1}{|\mathcal{R}|}\right)^{\frac{1}{2}} = \left(\frac{2\,a_1}{|\mathcal{R}|}\right)^{\frac{1}{2}} \qquad r_b = 2 \cdot \left(\frac{\lambda_2}{|\mathcal{R}|}\right)^{\frac{1}{2}} = \left(\frac{2\,a_2}{|\mathcal{R}|}\right)^{\frac{1}{2}}$$

Invariant moments

- Hu's moments: seven combinations of the normalized centeral moments.
- Invariant to translation, scaling and orientation.

$$H_1 = \bar{\mu}_{20} + \bar{\mu}_{02}$$

$$H_2 = (\bar{\mu}_{20} - \bar{\mu}_{02})^2 + 4\,\bar{\mu}_{11}^2$$

$$H_3 = (\bar{\mu}_{30} - 3\,\bar{\mu}_{12})^2 + (3\,\bar{\mu}_{21} - \bar{\mu}_{03})^2$$

$$H_4 = (\bar{\mu}_{30} + \bar{\mu}_{12})^2 + (\bar{\mu}_{21} + \bar{\mu}_{03})^2$$

$$H_5 = (\bar{\mu}_{30} - 3\,\bar{\mu}_{12}) \cdot (\bar{\mu}_{30} + \bar{\mu}_{12}) \cdot \left[(\bar{\mu}_{30} + \bar{\mu}_{12})^2 - 3(\bar{\mu}_{21} + \bar{\mu}_{03})^2\right] + (3\,\bar{\mu}_{21} - \bar{\mu}_{03}) \cdot (\bar{\mu}_{21} + \bar{\mu}_{03}) \cdot \left[3\,(\bar{\mu}_{30} + \bar{\mu}_{12})^2 - (\bar{\mu}_{21} + \bar{\mu}_{03})^2\right]$$

$$H_6 = (\bar{\mu}_{20} - \bar{\mu}_{02}) \cdot \left[(\bar{\mu}_{30} + \bar{\mu}_{12})^2 - (\bar{\mu}_{21} + \bar{\mu}_{03})^2\right] + 4\,\bar{\mu}_{11} \cdot (\bar{\mu}_{30} + \bar{\mu}_{12}) \cdot (\bar{\mu}_{21} + \bar{\mu}_{03})$$

$$H_7 = (3\,\bar{\mu}_{21} - \bar{\mu}_{03}) \cdot (\bar{\mu}_{30} + \bar{\mu}_{12}) \cdot \left[(\bar{\mu}_{30} + \bar{\mu}_{12})^2 - 3\,(\bar{\mu}_{21} + \bar{\mu}_{03})^2\right] + (3\,\bar{\mu}_{12} - \bar{\mu}_{30}) \cdot (\bar{\mu}_{21} + \bar{\mu}_{03}) \cdot \left[3\,(\bar{\mu}_{30} + \bar{\mu}_{12})^2 - (\bar{\mu}_{21} + \bar{\mu}_{03})^2\right]$$

Projections

- Horizontal and vertical projection of the shapes

$$P_{hor}(v_0) = \sum_{u=0}^{M-1} I(u, v_0) \qquad \text{for } 0 < v_0 < N$$

$$P_{ver}(u_0) = \sum_{v=0}^{N-1} I(u_0, v) \qquad \text{for } 0 < u_0 < M$$