Digital Imaging and Multimedia

Filters

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Outlines

- What are Filters
- Linear Filters
- Convolution operation
- Properties of Linear Filters
- Application of filters
- Nonlinear Filter
- Normalized Correlation and finding patterns in images
- Sources:
  - Burger and Burge "Digital Image Processing" Chapter 6
  - Forsyth and Ponce "Computer Vision a Modern approach"

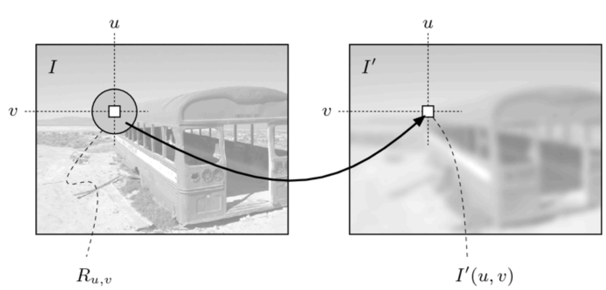
What is a Filter

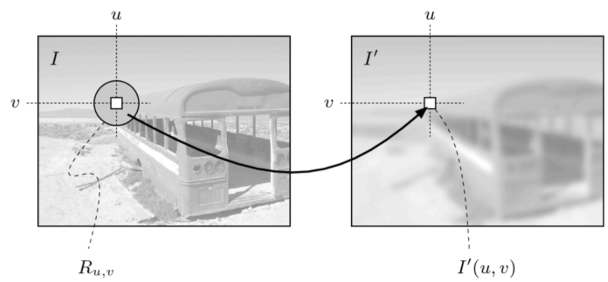
- Point operations are limited (why)
- They cannot accomplish tasks like sharpening or smoothing

Smoothing an image by averaging

- Replace each pixel by the average of its neighboring pixels
- Assume a 3x3 neighborhood:

$$I'(u,v) \leftarrow \frac{p_0 + p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8}{9}$$

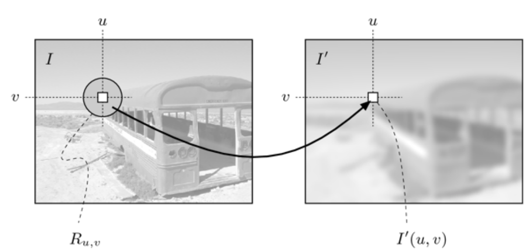




$$I'(u,v) \leftarrow \frac{p_0 + p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8}{9}$$

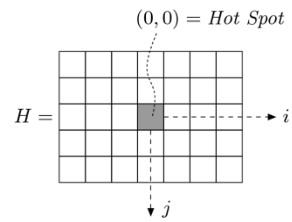
$$\begin{aligned} I'(u,v) \leftarrow \tfrac{1}{9} \cdot [\, & I(u-1,v-1) + I(u,v-1) + I(u+1,v-1) + \\ & I(u-1,v) + I(u,v) + I(u+1,v) + \\ & I(u-1,v+1) + I(u,v+1) + I(u+1,v+1)\,] \end{aligned}$$

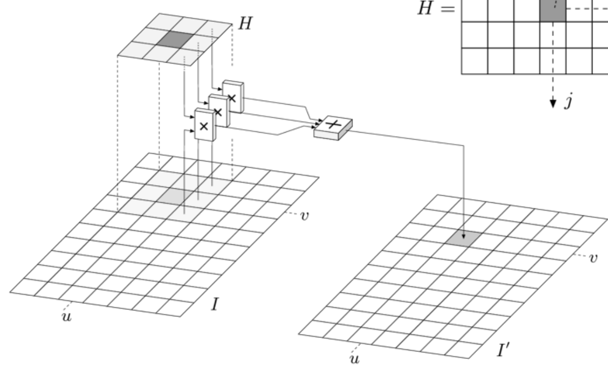
$$I'(u,v) \leftarrow \frac{1}{9} \cdot \sum_{j=-1}^{1} \sum_{i=-1}^{1} I(u+i, v+j)$$



- In general a filter applies a function over the values of a small neighborhood of pixels to compute the result
- The size of the filter = the size of the neighborhood: 3x3, 5x5, 7x7, …, 21x21,..
- The shape of the filter region is not necessarily square, can be a rectangle, a circle…
- Filters can be linear of nonlinear

Linear Filters: convolution





$$I'(u,v) \leftarrow \sum_{(i,j)\in R_H} I(u+i, v+j) \cdot H(i,j)$$

$$I'(u,v) \leftarrow \sum_{i=-1}^{i=1} \sum_{j=-1}^{j=1} I(u+i, v+j) \cdot H(i,j)$$

Averaging filter

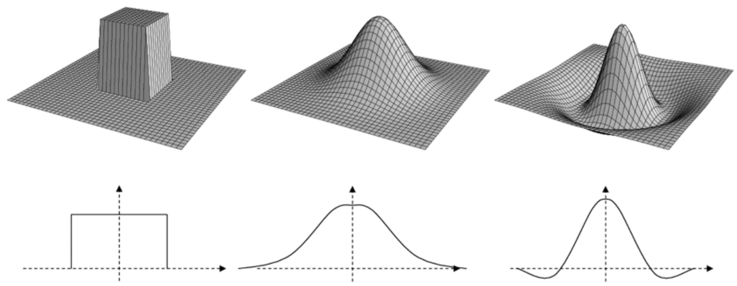
$$I'(u,v) \;\leftarrow\; \frac{p_0+p_1+p_2+p_3+p_4+p_5+p_6+p_7+p_8}{9}$$

$$\begin{aligned} I'(u,v) \;\leftarrow \tfrac{1}{9} \cdot [\, & I(u-1,v-1) && +I(u,v-1) && +I(u+1,v-1) && + \\ & I(u-1,v) && +I(u,v) && +I(u+1,v) && + \\ & I(u-1,v+1) && +I(u,v+1) && +I(u+1,v+1) \,] && \end{aligned}$$

$$H(i,j) \;=\; \begin{bmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \end{bmatrix} \;=\; \frac{1}{9} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$I'(u,v) \leftarrow \sum_{i=-1}^{i=1} \sum_{j=-1}^{j=1} I(u+i, v+j) \cdot H(i,j)$$

Types of Linear Filters



0	0	0	0	0
0	1	1	1	0
0	1	1	1	0
0	1	1	1	0
0	0	0	0	0

(a)

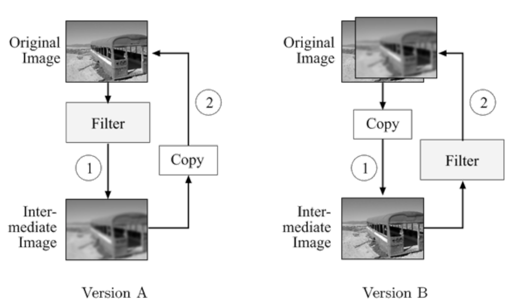
0	1	2	1	0
1	3	5	3	1
2	5	9	5	2
1	3	5	3	1
0	1	2	1	0

(b)

0	0	-1	0	0
0	-1	-2	-1	0
-1	-2	16	-2	-1
0	-1	-2	-1	0
0	0	-1	0	0

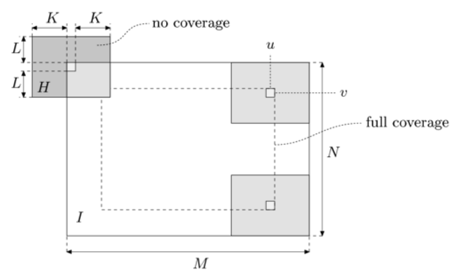
(c)

- Computing the filter operation
  - The filter matrix H moves over the original image I to compute the convolution operation
  - We need an intermediate image storage!
  - We need 4 for loops!
  - In general a scale is needed to obtain a normalized filter.
  - Integer coefficient is preferred to avoid floating point operations



- For a filter of size (2K+1) x (2L+1), if the image size is MxN, the filter is computed over the range:

$$K \leq u' \leq (M-K-1) \qquad \text{and} \qquad L \leq v' \leq (N-L-1)$$



Another smoothing filter

$$H(i,j) = \begin{bmatrix} 0.075 & 0.125 & 0.075 \\ 0.125 & \mathbf{0.2} & 0.125 \\ 0.075 & 0.125 & 0.075 \end{bmatrix}$$

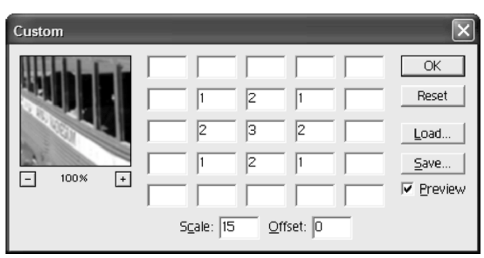
```
public void run(ImageProcessor orig) {
    int w = orig.getWidth();
    int h = orig.getHeight();
    // 3 × 3 filter matrix
    double[][] filter = {
        {0.075, 0.125, 0.075},
        {0.125, 0.200, 0.125},
        {0.075, 0.125, 0.075}
    };
    ImageProcessor copy = orig.duplicate();

    for (int v = 1; v <= h-2; v++) {
        for (int u = 1; u <= w-2; u++) {
            // compute filter result for position (u,v)
            double sum = 0;
            for (int j = -1; j <= 1; j++) {
                for (int i = -1; i <= 1; i++) {
                    int p = copy.getPixel(u+i, v+j);
                    // get the corresponding filter coefficient:
                    double c = filter[j+1][i+1];
                    sum = sum + c * p;
                }
            }
            int q = (int) Math.round(sum);
            orig.putPixel(u, v, q);
        }
    }
}
```

Integer coefficient

$$H(i,j) = \begin{bmatrix} 0.075 & 0.125 & 0.075 \\ 0.125 & \underline{0.200} & 0.125 \\ 0.075 & 0.125 & 0.075 \end{bmatrix} = \frac{1}{40} \begin{bmatrix} 3 & 5 & 3 \\ 5 & \underline{8} & 5 \\ 3 & 5 & 3 \end{bmatrix}$$

- Ex: linear filter in Adobe photoshop

$$I'(u,v) \leftarrow \text{Offset} + \frac{1}{\text{Scale}} \sum_{j=-2}^{j=2} \sum_{i=-2}^{i=2} I(u+i, v+j) \cdot H(i,j)$$

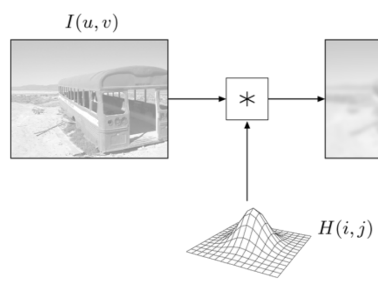


Mathematical Properties of Linear Convolution

- For any 2D discrete signal, convolution is defined as:

$$I'(u,v) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} I(u-i, v-j) \cdot H(i,j)$$

$$I' = I * H$$



Properties

- Commutativity

$$I * H = H * I$$

- Linearity

$$(s \cdot I) * H = I * (s \cdot H) = s \cdot (I * H)$$

$$(I_1 + I_2) * H = (I_1 * H) + (I_2 * H)$$

(notice) $(b + I) * H \neq b + (I * H)$

- Associativity

$$A * (B * C) = (A * B) * C$$

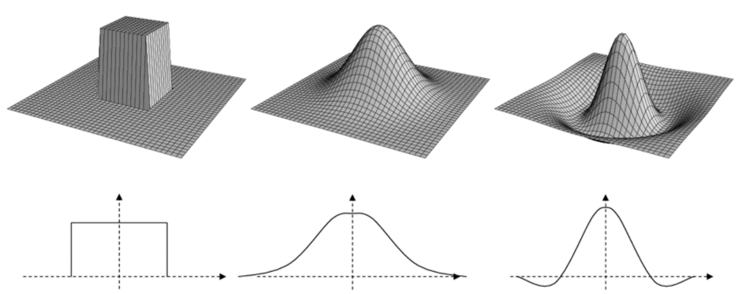
Properties

- Separability

$$H = H_1 * H_2 * \ldots * H_n$$

$$\begin{aligned} I * H &= I * (H_1 * H_2 * \ldots * H_n) \\ &= \big(\ldots((I * H_1) * H_2) * \ldots * H_n\big) \end{aligned}$$

Types of Linear Filters



0	0	0	0	0
0	1	1	1	0
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(a)

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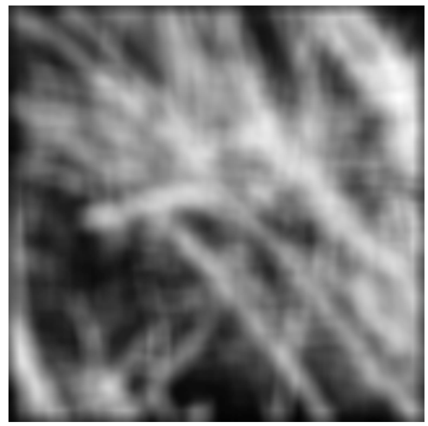
0	0	-1	0	0

(c)

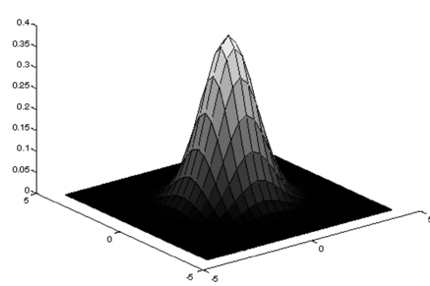
Smoothing by Averaging vs. Gaussian

Flat kernel: all weights equal 1/N



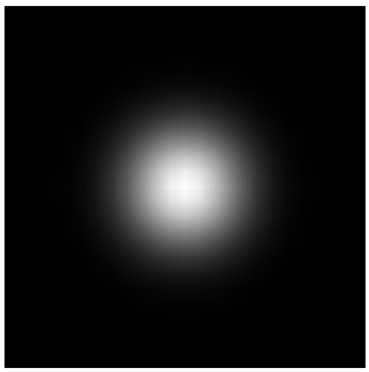
Smoothing with a Gaussian

- Smoothing with an average actually doesn’t compare at all well with a defocussed lens
  - Most obvious difference is that a single point of light viewed in a defocussed lens looks like a fuzzy blob; but the averaging process would give a little square.



- A Gaussian gives a good model of a fuzzy blob

An Isotropic Gaussian

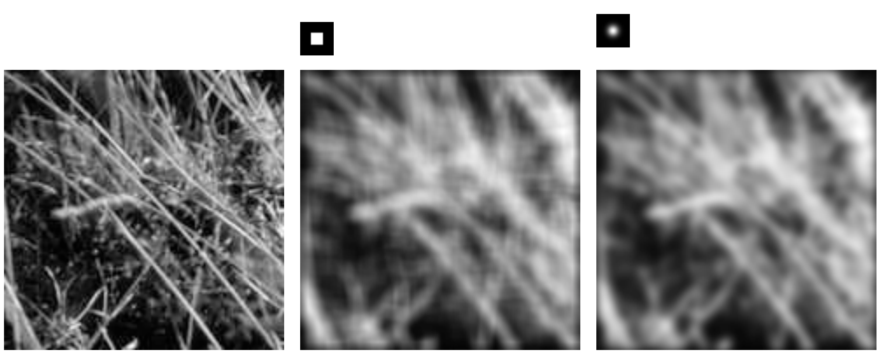


- The picture shows a smoothing kernel proportional to

$$\exp\left(-\left(\frac{x^2+y^2}{2\sigma^2}\right)\right)$$

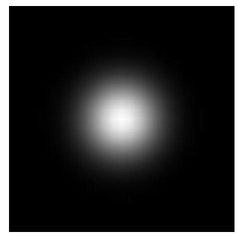
(which is a reasonable model of a circularly symmetric fuzzy blob)

Smoothing with a Gaussian



Gaussian smoothing

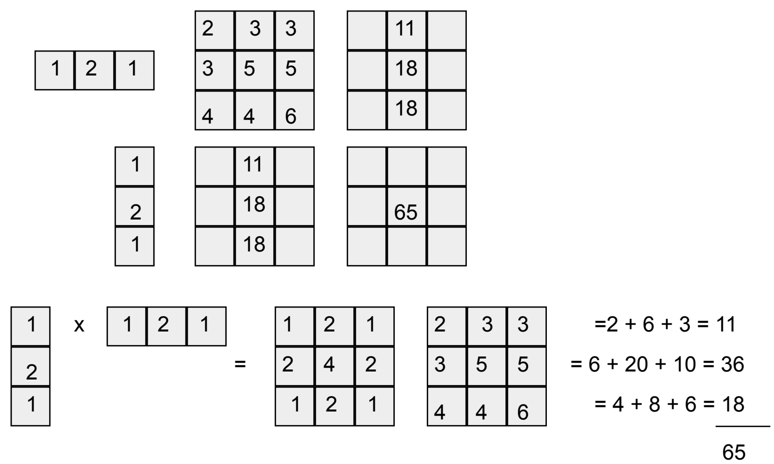
- Advantages of Gaussian filtering
  - rotationally symmetric (for large filters)
  - filter weights decrease monotonically from central peak, giving most weight to central pixels
  - Simple and intuitive relationship between size of σ and the smoothing.
  - The Gaussian is separable…



Advantage of seperability

- First convolve the image with a one dimensional horizontal filter
- Then convolve the result of the first convolution with a one dimensional vertical filter
- For a *k*x*k* Gaussian filter, 2D convolution requires k^2 operations per pixel
- But using the separable filters, we reduce this to *2k* operations per pixel.

Separability



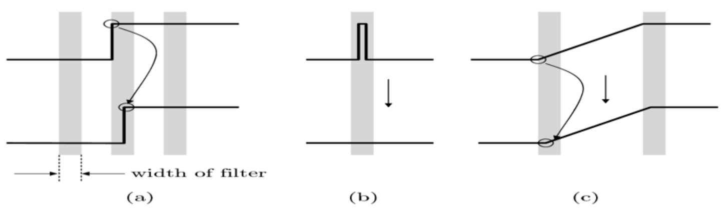
Advantages of Gaussians

- Convolution of a Gaussian with itself is another Gaussian
  - so we can first smooth an image with a small Gaussian
  - then, we convolve that smoothed image with another small Gaussian and the result is equivalent to smoother the original image with a larger Gaussian.
  - If we smooth an image with a Gaussian having sd σ twice, then we get the same result as smoothing the image with a Gaussian having standard deviation (2σ)

Nonlinear Filters

- Linear filters have a disadvantage when used for smoothing or removing noise: all image structures are blurred, the quality of the image is reduced.
- Examples of nonlinear filters:
  - Minimum and Maximum filters

$$I'(u,v) \leftarrow \min\{I(u+i,v+j) \mid (i,j) \in R\}$$
$$I'(u,v) \leftarrow \max\{I(u+i,v+j) \mid (i,j) \in R\}$$

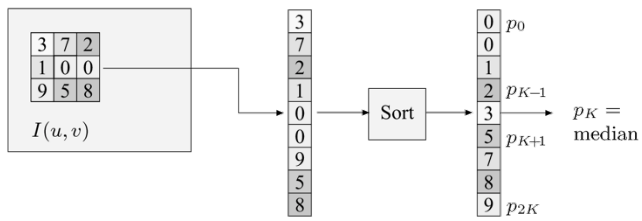


(a) (b) (c)

Median Filter

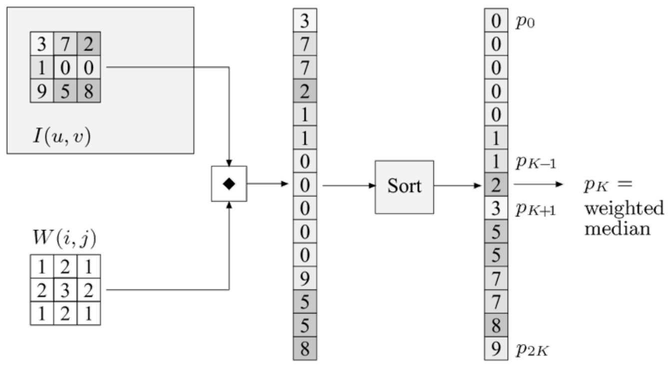
- Much better in removing noise and keeping the structures

$$I'(u,v) \leftarrow \text{median}\,\{I(u+i, v+j) \mid (i,j) \in R\}$$

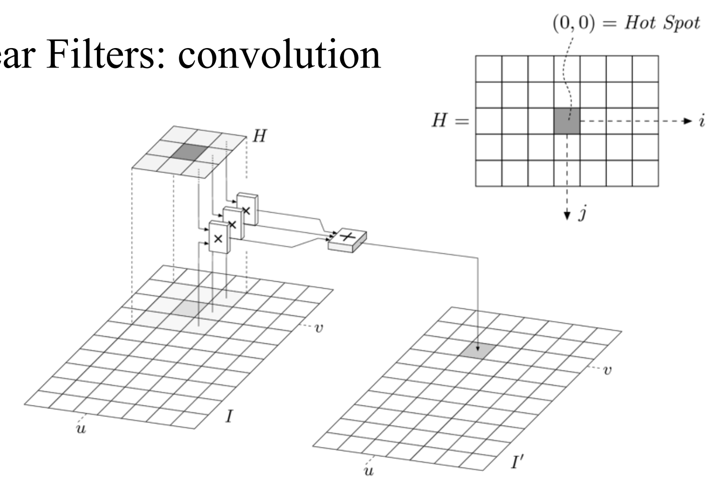


(a) (b) (c)

Weighted median filter



Linear Filters: convolution

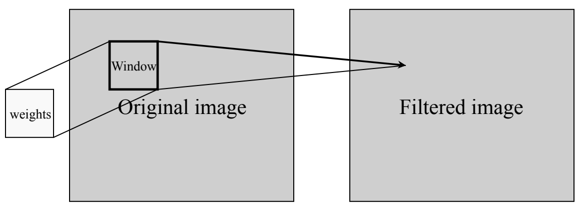


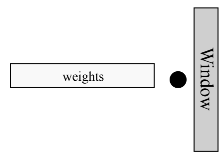
$$I'(u,v) \leftarrow \sum_{(i,j)\in R_H} I(u+i, v+j) \cdot H(i,j)$$

$$I'(u,v) \leftarrow \sum_{i=-1}^{i=1} \sum_{j=-1}^{j=1} I(u+i, v+j) \cdot H(i,j)$$

Convolution as a Dot Product

- Applying a filter at some point can be seen as taking a dot-product between the image and some vector
- Convoluting an image with a filter is equivalent to taking the dot product of the filter with each image window.

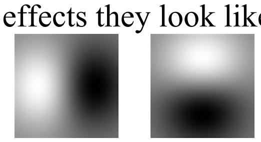


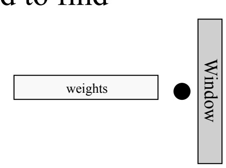


- Largest value when the vector representing the image is parallel to the vector representing the filter
- Filter responds most strongly at image windows that looks like the filter.
- Filter responds stronger to brighter regions! (drawback)

Insight:

- filters look like the effects they are intended to find
- filters find effects they look like



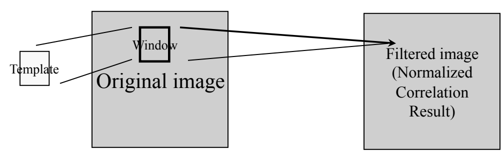


Ex: Derivative of Gaussian used in edge detection looks like edges

Normalized Correlation

- Convolution with a filter can be used to find templates in the image.
- Normalized correlation output is filter output, divided by root sum of squares of values over which filter lies
- Consider template (filter) *M* and image window *N*:

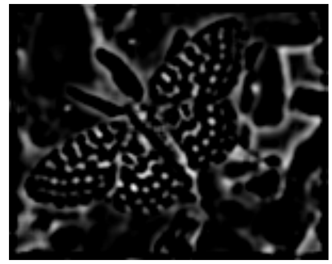
$$C = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} M(i,j)N(i,j)}{[\sum_{i=1}^{n}\sum_{j=1}^{n} M(i,j)^2 \sum_{i=1}^{n}\sum_{j=1}^{n} N(i,j)^2]^{1/2}}$$



Normalized Correlation

$$C = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} M(i,j)N(i,j)}{[\sum_{i=1}^{n}\sum_{j=1}^{n} M(i,j)^2 \sum_{i=1}^{n}\sum_{j=1}^{n} N(i,j)^2]^{1/2}}$$

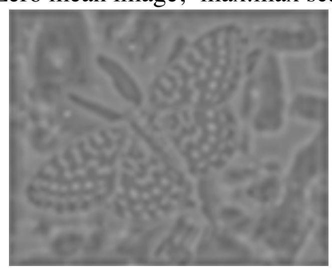
- This correlation measure takes on values in the range [0,1]
- it is 1 if and only if N = cM for some constant c
- so N can be uniformly brighter or darker than the template, M, and the correlation will still be high.
- The first term in the denominator, $\Sigma\Sigma M^2$ depends only on the template, and can be ignored
- The second term in the denominator, $\Sigma\Sigma N^2$ can be eliminated if we first normalize the grey levels of *N* so that their total value is the same as that of *M* - just scale each pixel in N by $\Sigma\Sigma M / \Sigma\Sigma N$



Positive responses

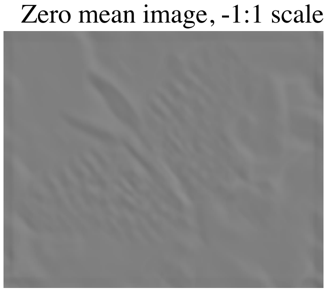
Zero mean image, -1:1 scale

Zero mean image, -max:max scale



Positive responses

Zero mean image, -1:1 scale



Zero mean image, -max:max scale

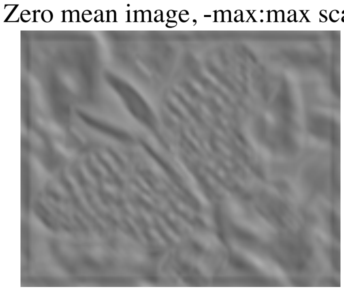


Figure from “Computer Vision for Interactive Computer Graphics,” W.Freeman et al, IEEE Computer Graphics and Applications, 1998 copyright 1998, IEEE