CS 205: Sample Final Exam – December 6th, 2004

1. **[10 points]** Let $A = \{1, 3, 5, 7, 9\}$, $B = \{4, 5, 6, 7, 8\}$, $C = \{2, 4, 6, 8, 10\}$, $D = \{1, 2, 3\}$ and let the universal set be $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

   (a) $\mid A \mid =$

   (b) $A \cap B =$

   (c) $\overline{B} =$

   (d) $(A \cup B) - C =$

   (e) $A - (B \cup C) =$

   (f) $(B \cap \overline{A}) \cup (B \cap \overline{C}) =$

   (g) $\overline{(A \cap C)} =$

   (h) $\overline{(A \cup C)} =$

   (i) $P(D) =$

   (j) $D \times (B \cap C) =$

   (k) $\mid P(A) \mid =$

   (l) Represent each of the sets A , B and C using bit strings. Then, use bit string representation and bitwise logical operations to find
   $(A \cup B) \cap (\overline{B} \cup C)$

2. (a) **[2 points]** find the dual the following propositions: $(q \wedge \neg p) \vee (p \wedge T) \vee (\neg q \wedge F)$

(b) **[3 points]** Put the following proposition into a disjunctive normal form: $p \wedge (q \oplus r)$

(c) **[5 points]** Let the universe of discourse consists of all students in your class and let $C(x)$ be "x has a cat ", let D(x) be " x has a dog ", and let $F(x)$ be " x has a ferret. " Express each of the following statements in terms of $C(x), D(x), F(x)$, quantifiers, and logical connectives.

i. A student in your class has a cat, a dog and a ferret.

ii. Some student has a cat and a dog but not a ferret.

iii. No student in your class has a cat, a dog, and a ferret.

iv. For each of the three animals, cats, dogs, and ferrets, there is a student in your class who has one of these animals as a pet.

v. None of the students who have cats have dogs.

3. **[10 points]** Let p, q, and r be the propositions

p: You get an A on the final exam

q: You do every exercise in this book

r: You get an A in this class

Write the following propositions using p, q, and r and logical connectives

(a) If you do every exercise in this book you will get an A in this class.

(b) You get an A in this class only if you get an A in the final exam.

(c) To get an A in this class it is necessary for you to get an A in the final

(d) Getting an A on the final and doing every exercise in this book is sufficient for getting an A in this class.

(e) You will get an A in this class if and only if you either do every exercise in this book or you get an A on the final.

4. **[10 points]** Determine the truth value of each of the following statements if the universe of discourse is the set of real numbers (explain your answers) .

(a) $\forall x \exists y (x^2 = y)$

(b) $\forall x \exists y (y^2 = x)$

(c) $\exists x \forall y (xy = 0)$

(d) $\forall x (x \neq 0 \rightarrow \exists y (xy = 1))$

(e) $\exists x \forall y (y \neq 0 \rightarrow xy = 1)$

(f) $\exists x \exists y ((x + 2y = 2) \wedge (2x + 4y = 5))$

(g) $\forall x \exists y (y^2 - x = 1)$

(h) $\forall x \forall y \exists z (z = (x + y)/2)$

(i) $\exists ! x (x^2 = 1)$

(j) $\exists ! x \neg P(x) \rightarrow \neg \forall x P(x)$

5. **[14 points]** Consider the following relations on the set of all real numbers. Determine whether each of them is reflexive, symmetric, antisymmetric or transitive. Fill in your answers as "yes" or "no" in the table provided.

Relation	Reflexive ?	Symmetric?	Antisymmetric ?	Transitive ?
$x + y = 0$				
$x = 2y$				
$xy = 0$				
$xy \geq 1$				
$x \neq y$				
x is multiple of y				
$x = y^2$				

6. **[12 points]** Let R and S be relations on the set $\{1, 2, 3, 4\}$ where $R = \{(1, 2), (1, 3), (2, 3), (2, 4), (3, 1)\}$ and $S = \{(2, 1), (3, 1), (3, 2), (4, 2)\}$. Find each of the following:

   (a) r(R) (the reflexive closure of R)
   (write your final answer in zero-one matrix form)

   (b) s(S) (the symmetric closure of S)
   (write your final answer in zero-one matrix form)

   (c) The composite of R and S, i.e., $S \circ R =$ **(write your final answer in zero-one matrix form)**

   (d) The composite of S and R, i.e., $R \circ S =$ **(write your final answer in zero-one matrix form)**

(e) R^2, R^3, R^4 **(write your final answer in zero-one matrix form)**

(f) Find the transitive closure of R **(write your final answer in zero-one matrix form)**

7. [**10 points**] Let A be the set of integers and let R be the relation on $A \times A$ defined by: $(a, b)R(c, d)$ iff $a + d = b + c$

(a) Prove that R is an equivalence relation.

(b) Now let $A = \{1, 2, 3, \cdots, 14, 15\}$ and let R be defined on $A \times A$ as above. Find the equivalence class of (2,7)

8. **[10 points]**

(a) Prove that the inclusion relation $\subseteq$ is a partial ordering on the power set of a set S.

(b) Is $(P(S), \subseteq)$ totally ordered ? explain why.

(c) Let $S = \{a, b, c\}$ Draw a Hasse diagram for $(P(S), \subseteq)$.

9. (a) **[5 points]** Prove that if n is an integer and $n^3 + 5$ is odd then n is even

(b) **[5 points]** Prove that n^2 is divisible by a prime p **if and only if** n is divisible by p.

10. (a) **[5 points]** Prove that $6 - 7\sqrt{2}$ is irrational. (you can use the theorem: $\sqrt{2}$ is irrational)

(b) **[5 points]** Prove or disprove that if r is any rational number and s is any irrational number, then r/s is irrational

11. (a) **[6 points]** Use mathematical induction to prove that for all integers $n \geq 0$, $2^{3n} - 1$ is divisible by 7.

(b) **[6 points]** Use mathematical induction to prove that

$$\sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2$$

for all integers $n \geq 0$

12. **[10 points]** Determine whether 1011 belongs to each of these regular sets:

(a) 10^*1^*

(b) $1(01)^*1^*$

(c) $(10)^*(11)^*$

(d) $(10)^*1011$

(e) $(1 \cup 00)(01 \cup 0)1^*$