Section 6.5
Equivalence Relations

Now we group properties of relations together to define new types of important relations.

Definition: A relation R on a set A is an *equivalence relation* iff R is

- reflexive
- symmetric

and

- transitive

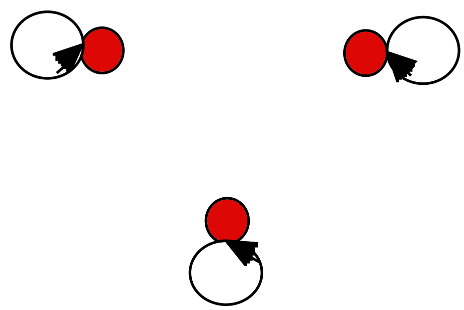
It is easy to recognize equivalence relations using digraphs.

- The subset of all elements related to a particular element forms a universal relation (contains all possible arcs) on that subset. The (sub)digraph representing the subset is called a *complete* (sub)digraph. All arcs are present.

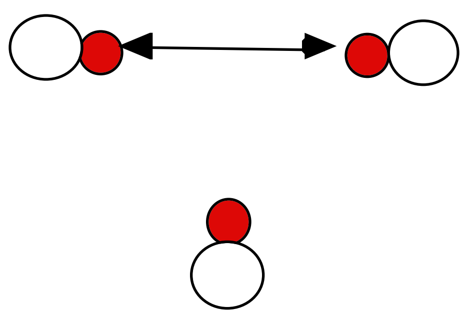
- The number of such subsets is called the *rank* of the equivalence relation

Examples:

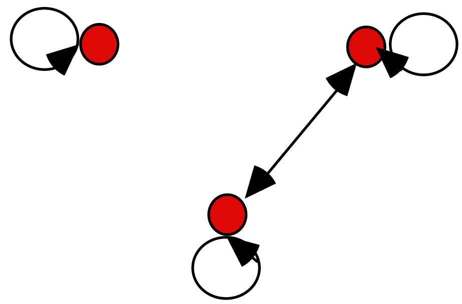
A has 3 elements:



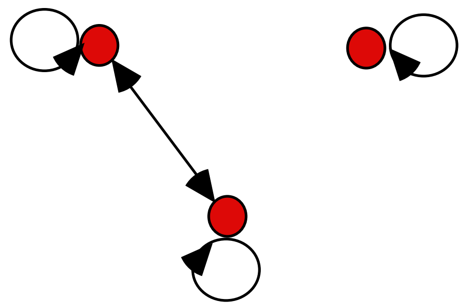
rank = 3



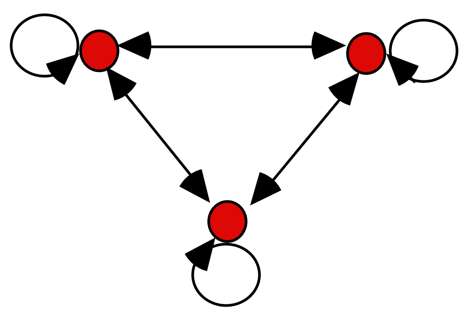
rank = 2



rank = 2



rank = 2



rank = 1

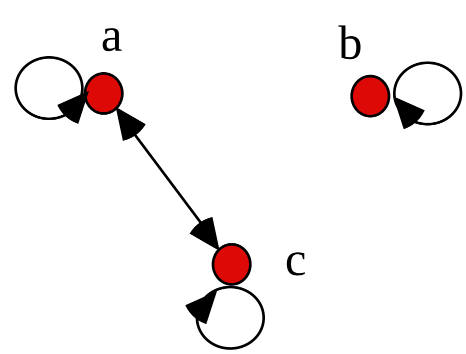
• Each of the subsets is called an *equivalence class.*

• A bracket around an element means the equivalence class in which the element lies.

[x] = {y | <x, y> is in R}

• The element in the bracket is called a *representative* of the equivalence class. We could have chosen any one.

Examples:



[a] = {a, c}, [c] = {a, c}, [b] = {b}.
rank = 2

An interesting counting problem:

Count the number of equivalence relations on a set A with n elements. Can you find a recurrence relation?

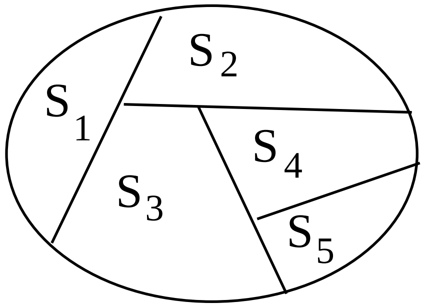
The answers are

- 1 for $n = 1$
- 3 for $n = 2$
- 5 for $n = 3$

How many for$n = 4$?

Definition: Let $S_1, S_2, . . ., S_n$ be a collection of subsets of A. Then the collection forms a *partition* of A if the subsets are nonempty, disjoint and *exhaust* A:

- $S_i \neq$
- $S_i \cap S_j =$ if $i \neq j$
- $\bigcup S_i = A$



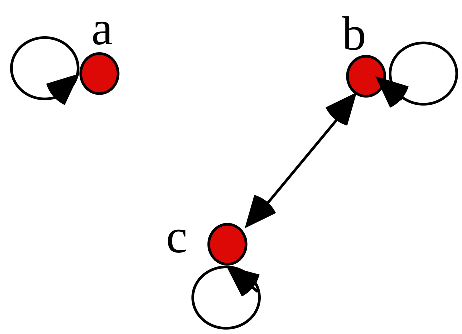
Theorem: The equivalence classes of an equivalence relation R *partition* the set A into disjoint nonempty subsets whose union is the entire set.

This partition is denoted *A/R* and called

- the *quotient set,* or
- *the partition of A induced by R,* or,
- *A modulo R.*

Examples:

- $A \times A$
- $A =$
-



$$A = [a] \quad [b] = [a] \quad [c] = \{a\} \quad \{b,c\}$$

rank = 2

Theorem: Let R be an equivalence relation on A. Then

either

$$[a] = [b]$$

or

$$[a] \cap [b] =$$

Theorem: If R_1 and R_2 are equivalence relations on A then $R_1 \cap R_2$ is an equivalence relation on A.

Proof: It suffices to show that the intersection of

- reflexive relations is reflexive,

- symmetric relations is symmetric,

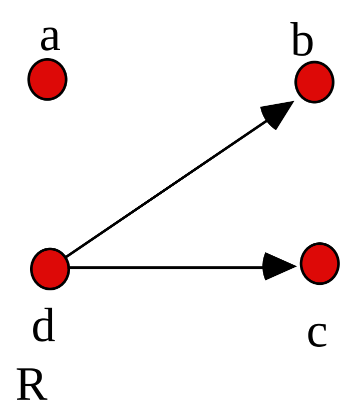
and

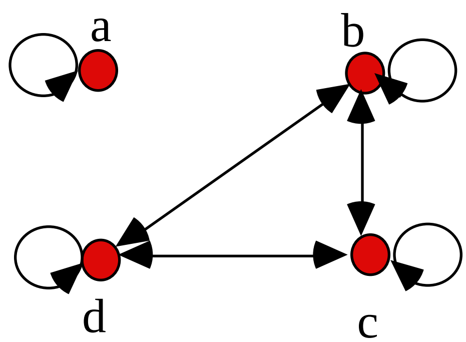
- transitive relations is transitive.

You provide the details.

Definition: Let R be a relation on A. Then the reflexive, symmetric, transitive closure of R, tsr(R), is an equivalence relation on A, called the *equivalence relation induced by R.*

Example:





tsr(*R*)
rank = 2
A = *[a]*   *[b]* = *{a}*   *{b, c, d}*

A/R = *{{a}, {b, c, d}}*

Theorem: tsr(*R*) is an equivalence relation

Proof:

We have to be careful and show that tsr(*R*) is still symmetric and reflexive.

• Since we only add arcs vs. deleting arcs when computing closures it must be that tsr(*R*) is reflexive since all loops <x, x> on the diagraph must be present when constructing r(*R*).

• If there is an arc <x, y> then the symmetric closure of r(*R*) ensures there is an arc <y, x>.

• Now argue that if we construct the transitive closure of sr(R) and we add an edge <x, z> because there is a path from x to z, then there must also exist a path from z to x (why?) and hence we also must add an edge <z, x>. Hence the transitive closure of sr(R) is symmetric.

Q. E. D.