

## Section 1.3 Predicates and Quantifiers

---

A generalization of propositions - *propositional functions* or *predicates*.: propositions which contain variables

Predicates become propositions once every variable is *bound*- by

U • assigning it a value from the *Universe of Discourse*

or

- quantifying it
- 

Examples:

Let  $U = \mathbb{Z}$ , the integers =  $\{ \dots -2, -1, 0, 1, 2, 3, \dots \}$

•  $P(x): x > 0$  is the predicate. It has no truth value until the variable  $x$  is bound.

Examples of propositions where  $x$  is assigned a value:

- $P(-3)$  is false,
- $P(0)$  is false,
- $P(3)$  is true.

The collection of integers for which  $P(x)$  is true are the positive integers.

---

•  $P(y) \wedge \neg P(0)$  is not a proposition. The variable  $y$  has not been bound. However,  $P(3) \wedge \neg P(0)$  is a proposition which is true.

• Let  $R$  be the three-variable predicate  $R(x, y, z): x + y = z$

Find the truth value of

$R(2, -1, 5), R(3, 4, 7), R(x, 3, z)$

---

---

## Quantifiers

- **Universal**

$P(x)$  is true for every  $x$  in the universe of discourse.

Notation: *universal quantifier*

$$\forall x P(x)$$

‘For all  $x$ ,  $P(x)$ ’, ‘For every  $x$ ,  $P(x)$ ’

The variable  $x$  is bound by the universal quantifier producing a proposition.

---

Example:  $U = \{1, 2, 3\}$

$$\forall x P(x) \quad P(1) \quad P(2) \quad P(3)$$

---

- **Existential**

$P(x)$  is true for some  $x$  in the universe of discourse.

Notation: *existential quantifier*

$$\exists x P(x)$$

‘There is an  $x$  such that  $P(x)$ ,’ ‘For some  $x$ ,  $P(x)$ ,’ ‘For at least one  $x$ ,  $P(x)$ ,’ ‘I can find an  $x$  such that  $P(x)$ .’

---

Example:  $U = \{1, 2, 3\}$   
 $xP(x)$      $P(1)$      $P(2)$      $P(3)$

---

• **Unique Existential**

$P(x)$  is true for one and only one  $x$  in the universe of discourse.

Notation: *unique existential quantifier*

$$\exists! x P(x)$$

‘There is a unique  $x$  such that  $P(x)$ ,’ ‘There is one and only one  $x$  such that  $P(x)$ ,’ ‘One can find only one  $x$  such that  $P(x)$ .’

---

Example:

$U = \{1, 2, 3\}$

**Truth Table:**

P(1)	P(2)	P(3)	$\neg xP(x)$
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	0

How many minterms are in the DNF?

Note:

**REMEMBER!**

A predicate is not a proposition until *all* variables have been bound either by quantification or assignment of a value!

---

Equivalences involving the negation operator

$$\neg \neg xP(x) \quad x \neg \neg P(x)$$

$$\neg \neg xP(x) \quad x \neg \neg P(x)$$

Distributing a negation operator across a quantifier changes a universal to an existential and vice versa.

---

Multiple Quantifiers: read left to right . . .

---

Example: Let  $U = \mathbb{R}$ , the real numbers,

$P(x,y): xy=0$

$$\forall x \exists y P(x,y)$$

$$\forall x \forall y P(x,y)$$

$$\exists x \forall y P(x,y)$$

$$\exists x \exists y P(x,y)$$

The only one that is false is the first one.

Suppose  $P(x,y)$  is the predicate  $x/y=1$ ?

---

Example:

Let  $U = \{1,2,3\}$ . Find an expression equivalent to

$$\forall x \exists y P(x,y)$$

where the variables are bound by substitution instead:

Expand from inside out or outside in.

Outside in:

$$\begin{array}{ccc} yP(1, y) & yP(2, y) & yP(3, y) \\ [P(1,1) & P(1,2) & P(1,3)] \\ [P(2,1) & P(2,2) & P(2,3)] \\ [P(3,1) & P(3,2) & P(3,3)] \end{array}$$

---

## Converting from English

(can be very difficult)

---

Examples:

F(x): x is a fleegle

S(x): x is a snurd

T(x): x is a thingamabob

$U = \{\text{fleegles, snurds, thingamabobs}\}$

(Note: the equivalent form using the existential quantifier is also given)

- Everything is a fleegle

$$\exists x F(x)$$

$$\neg \exists x \neg F(x)$$

- Nothing is a snurd.

$$\begin{aligned} & x \neg S(x) \\ & \neg \exists x S(x) \end{aligned}$$

- All fleegles are snurds.

$$\begin{aligned} & \forall x [F(x) \rightarrow S(x)] \\ & \forall x [\neg F(x) \vee S(x)] \\ & \forall x \neg [F(x) \wedge \neg S(x)] \\ & \neg \exists x [F(x) \wedge \neg S(x)] \end{aligned}$$

- Some fleegles are thingamabobs.

$$\begin{aligned} & \exists x [F(x) \wedge T(x)] \\ & \neg \forall x [\neg F(x) \vee \neg T(x)] \end{aligned}$$

- No snurd is a thingamabob.

$$\begin{aligned} & \forall x [S(x) \rightarrow \neg T(x)] \\ & \neg \exists x [S(x) \wedge T(x)] \end{aligned}$$

- If any fleegle is a snurd then it's also a thingamabob

$$\begin{aligned} & \exists x [(F(x) \wedge S(x)) \wedge T(x)] \\ & \neg \forall x [F(x) \wedge S(x) \rightarrow \neg T(x)] \end{aligned}$$

---

## Extra Definitions:

- An assertion involving predicates is *valid* if it is true for every universe of discourse.
- An assertion involving predicates is *satisfiable* if there is a universe and an interpretation for which the assertion is true. Else it is *unsatisfiable*.
- The *scope* of a quantifier is the part of an assertion in which variables are bound by the quantifier

---

## Examples:

Valid:  $\forall x \neg S(x) \quad \neg \exists x S(x)$

Not valid but satisfiable:  $\exists x [F(x) \wedge T(x)]$

Not satisfiable:  $\exists x [F(x) \wedge \neg F(x)]$

Scope:  $\exists x [F(x) \wedge S(x)]$  vs.  $\exists x [F(x)] \wedge \exists x [S(x)]$

---

## Dangerous situations:

- Commutativity of quantifiers

$\forall x \exists y P(x, y) \quad \exists y \forall x P(x, y)?$   
YES!

$\exists x \forall y P(x, y) \quad \forall y \exists x P(x, y)?$   
NO!

DIFFERENT MEANING!

- Distributivity of quantifiers over operators

$\forall x [P(x) \wedge Q(x)] \quad \forall x P(x) \wedge \forall x Q(x)?$   
YES!

$\forall x [P(x) \vee Q(x)] \quad \forall x [P(x) \vee Q(x)]?$   
NO!

---