Uniform Derandomization from Pathetic Lower Bounds

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May 4, 2012

Abstract

The notion of probabilistic computation dates back at least to Turing, and he also wrestled with the practical problems of how to implement probabilistic algorithms on machines with, at best, very limited access to randomness. A more recent line of research, known as derandomization, studies the extent to which randomness is superfluous. A recurring theme in the literature on derandomization is that probabilistic algorithms can be simulated quickly by *deterministic* algorithms, if one can obtain *impressive* (i.e., superpolynomial, or even nearly-exponential) circuit size lower bounds for certain problems. In contrast to what is needed for derandomization, existing lower bounds seem rather pathetic (linear-size lower bounds for general circuits [IM02], nearly cubic lower bounds for formula size [Hås98], nearly quadratic size lower bounds for branching programs [Nec66], n^{1+c_d} for depth d threshold circuits [IPS97]). Here, we present two instances where "pathetic" lower bounds of the form $n^{1+\epsilon}$ would suffice to derandomize interesting classes of probabilistic algorithms.

We show:

- If the word problem over S_5 requires constant-depth threshold circuits of size $n^{1+\epsilon}$ for some $\epsilon > 0$, then any language accepted by uniform polynomial-size probabilistic threshold circuits can be solved in subexponential time (and more strongly, can be accepted by a uniform family of deterministic constant-depth threshold circuits of subexponential size.)
- If there are no constant-depth arithmetic circuits of size $n^{1+\epsilon}$ for the problem of multiplying a sequence of n 3-by-3 matrices, then for every constant d, black-box identity testing for depth-d arithmetic circuits with bounded individual degree can be performed in subexponential time (and even by a uniform family of deterministic constant-depth AC^0 circuits of subexponential size).

*Supported in part by NSF Grants CCF-0830133, CCF-0832787, and CCF-1064785. Some of this work was performed while this author was a visiting scholar at the University of Cape Town.

†Supported in part by NSF Grants CCF-0830133, CCF-0832787 and CCF-1064785.

1 Introduction

Alan Turing was interested in probabilistic computation, on both practical and theoretical levels. Already in 1950, he explicitly proposed the notion of extending deterministic computing devices by providing access to a "random element" [Tur50]. Note that this was roughly contemporaneous with the development of Monte-Carlo methods [MU49]. Turing also participated in the development of the Mark I computer project at Manchester University, and according to some sources [CK80] he pushed for the inclusion of a random noise generator at the machine instruction level (although this feature was not successful). Thus, from almost the very beginning of the mathematical study of computation, there has been interest in probabilistic computation, as well as an appreciation of the obstacles that lie in the way of practical implementations of probabilistic algorithms.

One promising avenue for dealing with the scarcity of truly random bits is to show that, in many cases, there is no reason to use randomness at all. Hardness-based derandomization is one of the success stories of the past quarter century. The main thread of this line of research dates back to the work of Shamir, Yao, and Blum and Micali [Sha83, Yao82, BM84], and involves showing that, given a suitably hard function f, one can construct pseudorandom generators and hitting-set generators. Much of the progress on this front over the years has involved showing how to weaken the hardness assumption on f and still obtain useful derandomizations [BFNW93], [AK01], [IW97], [IW01], [KvM02], [ACR99], [ACR98], [ACRT99], [BF99], [MV05], [GW99], [GVW], [ISW06], [STV01], [SU05], [Uma03]. In rare instances, it has been possible to obtain *unconditional* derandomizations using this framework; Nisan and Wigderson showed that uniform families of probabilistic AC^0 circuits can be simulated by uniform deterministic AC^0 circuits of size $n^{\log^{O(1)} n}$ [Nis91], [NW94], [Vio05]. More often, the derandomizations that have been obtained are conditional, and rely on the existence of functions f that are hard on average. For certain large complexity classes $\mathcal{C}$ (notably including $\#\mathsf{P}$, PSPACE, and exponential time), various types of random self-reducibility and hardness amplification have been employed to show that such hard-on-average functions f exist in $\mathcal{C}$ if and only if there is some problem in $\mathcal{C}$ that requires large Boolean circuits [BFNW93, IW97].

A more recent thread in the derandomization literature has studied the implications of *arithmetic* circuit lower bounds for derandomization. Kabanets and Impagliazzo showed that, if the Permanent requires large *arithmetic circuits*, then the probabilistic algorithm to test if two arithmetic *formulae* (or more generally, two arithmetic circuits of polynomial degree) are equivalent can be simulated by a quick deterministic algorithm [KI04]. Subsequently, Dvir, Shpilka, and Yehudayoff built on the techniques of Kabanets and Impagliazzo, to show that if one could present a multilinear polynomial (such as the permanent) that requires depth d arithmetic formulae of size 2^{n^ϵ}, then the probabilistic algorithm to test if two arithmetic circuits of depth $d-5$ are equivalent (where in addition, the variables in these circuits have degree at most $\log^{O(1)} n$) can be derandomized to obtain a $2^{\log^{O(1)} n}$ deterministic algorithm for the problem [DSY09].

In this paper, we combine these two threads of derandomization with the recent insight that, in some cases, extremely modest-sounding (or even "pathetic") lower bounds can be amplified to obtain superpolynomial bounds [AK10]. In order to carry out this combination, we need to identify and exploit some special properties of certain functions in and near NC^1.

- The word problem over S_5 is one of the standard complete problems for NC^1 [Bar89]. Many of the most familiar complete problems for NC^1 have very efficient *strong downward self-reductions* [AK10]. We show that the word problem over S_5, in addition, is *randomly self-reducible*. (This was observed previously by Goldwasser *et al.* [GGH$^+$08].) This enables us to transform a "pathetic" *worst-case* size lower bound of $n^{1+\epsilon}$ on constant-depth threshold circuits, to a superpolynomial size *average-case* lower bound for this class of circuits. In turn, by making some adjustments to the Nisan-Wigderson generator, this average-case hard function can be used to give uniform subexponential derandomizations of probabilistic TC^0 circuits.

- Iterated Multiplication of n three-by-three matrices is a multilinear polynomial that is complete for arithmetic

NC^1 [BOC92]. In the Boolean setting, this function is strongly downward self-reducible via self-reductions computable in TC^0 [AK10]. Here we show that there is a corresponding *arithmetic* self-reduction; this enables us to amplify a lower bound of size $n^{1+\epsilon}$ for constant-depth arithmetic circuits, to obtain a super-polynomial lower bound for constant-depth arithmetic circuits. Then, by building on the approach of Dvir *et al.* [DSY09], we are able to obtain subexponential derandomizations of the identity testing problem for a class of constant-depth arithmetic circuits.

The rest of the paper is organized as follows: In Section 2, we give some preliminary definitions and notation. In Section 3, we show how to convert a modest worst-case hardness assumption to a strong average-case hardness separation of NC^1 from TC^0. We also present slightly weaker worst-case-to-average-case reductions for L and for the classes GapL and $GapNC^1$. In Section 4, we build on Section 3.1 to give a uniform derandomization of probabilistic TC^0 circuits. Finally, in Section 5 we prove our derandomization of a special case of polynomial identity testing under a modest hardness assumption.

2 Preliminaries

This paper will mainly discuss NC^1 and its subclass TC^0. The languages in NC^1 are accepted by families of circuits of depth $O(\log n)$ that are built with fan-in two AND and OR gates, and NOT gates of fan-in one. For any function $s(n)$, $TC^0(s(n))$ consists of languages that are decided by constant-depth circuit families of size at most $s(n)$ which contain only unbounded fan-in MAJORITY gates as well as unary NOT gates. $TC^0 = \cup_{k\geq 0} TC^0(n^k)$. $TC^0(\mathsf{SUBEXP}) = \cap_{\delta \geq 0} TC^0(2^{n^\delta})$. The definitions of $AC^0(s(n)), AC^0$, and $AC^0(\mathsf{SUBEXP})$ are similar, although MAJORITY gates are not allowed, and unbounded fan-in AND and OR gates are used instead.

We allow circuits to accept inputs not only from the Boolean alphabet $\{0,1\}$, but from any finite alphabet Σ. This is done by having σ-input gates for each $\sigma \in \Sigma$; a σ-input gate g connected to input symbol x_i evaluates to 1 if $x_i = \sigma$, and evaluates to 0 otherwise.

As is usual in arguments in derandomization based on the hardness of some function f, we require not only that f not have small circuits in order to be considered "hard", but furthermore we require that f needs large circuits at *every* relevant input length. This motivates the following definition.

Definition 1 *Let A be a language, and let D_A be the set $\{n : A \cap \Sigma^n \neq \emptyset\}$. We say that $A \in$ io-$TC^0(s(n))$ if there is an infinite set $I \subseteq D_A$ and a language $B \in TC^0(s(n))$ such that, for all $n \in I$, $A_n = B_n$ (where, for a language C, we let C_n denote the set of all strings of length n in C). Similarly, we define* io-TC^0 *to be* $\cup_{k\geq 0}$io-$TC^0(n^k)$.

Thus A requires large threshold circuits on *all* relevant input lengths if $A \notin$ io-TC^0. (A peculiarity of this definition is that if A is a *finite* set, or A_n is empty for infinitely many n, then $A \notin$ io-TC^0. This differs starkly from most notions of "io" circuit complexity that have been considered, but it allows us to consider "complex" sets A that are empty on infinitely many input lengths; the alternative would be to consider artificial variants of the "complex" sets that we construct, having strings of every length.)

Probabilistic circuits take an input divided into two pieces, the actual input and the random inputs. We say an input x is accepted by such a circuit C with probability p if, with respect to the uniform distribution U_D over the domain D from which the random inputs are drawn, $Pr_{r\sim U_D}[C(x,r) = 1] \geq p$. We say that input x is rejected by C with probability p if $Pr_{r\sim U_D}[C(x,r) = 0] \geq p$. When defining probabilistic complexity classes (such as probabilistic NC^1), we restrict the random inputs to come from $\{0,1\}^*$, and we say that a family of probabilistic circuits C_n accepts A if, for all $x \in A_n$, C_n accepts x with probability $\frac{2}{3}$, and for all other x, C_n rejects x with probability $\frac{2}{3}$.

The standard uniformity condition for small complexity classes is called DLOGTIME-uniformity. In order to provide its proper definition, we need to mention the *direct connection language* associated with a circuit family.

Definition 2 *Let $\mathcal{C} = (C_n)_{n\in\mathbb{N}}$ be a circuit family. The direct connection language L_{DC} of $\mathcal{C}$ is the set of all tuples having either the form $\langle n, p, q, b\rangle$ or $\langle n, p, d\rangle$, where*

- *If $q = \epsilon$, then b is the type of gate p in C_n;*
- *If q is the binary encoding of k, then b is the kth input to p in C_n.*
- *The gate p has fan-in d in C_n.*

The circuit family $\mathcal{C}$ is DLOGTIME-uniform if there is a deterministic Turing machine that accepts L_{DC} in linear time. For any circuit complexity class C, uC is its uniform counterpart, consisting of languages that are accepted by DLOGTIME-uniform circuit families. For more background on circuit complexity, we refer the reader to the textbook by Vollmer [Vol99]. The term "uniform derandomization" in the title refers to the fact that we are presenting uniform circuit families that compute derandomized algorithms; this should not be confused with doing derandomization based on uniform hardness assumptions.

The classes NC^1, GapL, and GapNC^1 all have complete problems under AC^0-Turing reducibility. (See [Vol99] for definitions of these terms.) All references to "completeness" refer to this notion of reducibility.

A particularly important complete language for NC^1 is the word problem WP for S_5, where S_5 is the symmetric group over 5 distinct elements [Bar89]. The input to the word problem is a sequence of permutations from S_5 and it is accepted if and only if the product of the sequence evaluates to the identity permutation. The corresponding *search* problem FWP is required to output the exact result of the iterated multiplication. A closely related *balanced* language is BWP, which stands for Balanced Word Problem.

Definition 3 *The input to* BWP *is a pair $\langle w_1 w_2..w_n, S\rangle$, where $\forall i \in [1..n],\ w_i \in S_5$, $S \subseteq S_5$ and $|S| = 60$. The pair $\langle w_1 w_2..w_n, S\rangle$ is in* BWP *if and only if $\Pi_{i=1}^n w_i \in S$.*

It is easy to verify that BWP is complete for NC^1 as well.

In the following sections, let FWP_n be the sub-problem of FWP where the domain is restricted to inputs of length n and let BWP_n be $\mathrm{BWP} \cap \{\langle \phi, S\rangle \mid \phi \in S_5^n,\ S \subseteq S_5,\ |S| = 60\}$. Note that BWP_n accepts exactly half of the instances in $\{\langle \phi, S\rangle \mid \phi \in S_5^n,\ S \subseteq S_5,\ |S| = 60\}$ since $|S_5| = 120$.

The following simplified version of Chernoff's bound turns out to be useful in our application.

Lemma 4 (Chernoff's bound) *Let $X_1, .., X_m$ be i.i.d. 0-1 random variables with $E[X_i] = p$. Let $X = \Sigma_{i=1}^m X_i$. Then for any $0 < \delta \leq 1$,*

$$Pr[X < (1-\delta)pm] \leq e^{-\frac{\delta^2 pm}{2}}.$$

3 The existence of an average-case hard language

3.1 Worst-case to Average-case Reduction for NC^1

In this section, we use random self-reducibility to show that, if $\mathrm{NC}^1 \neq \mathrm{TC}^0$, then there are problems in NC^1 that are hard on average for TC^0. First we recall the definition of hardness on average for decision problems.

Definition 5 *Let U_D denote the uniform distribution over all inputs in a finite domain D. For any Boolean function $f : D \to \{0, 1\}$, f is $(1-\epsilon)$-hard for a set of circuits S, if, for every $C \in S$, we have that $Pr_{x\sim U_D}[f(x) = C(x)] < 1 - \epsilon$.*

We will sometimes abuse notation by identifying a set with its characteristic function. For languages to be considered hard on average, we consider only those input lengths where the language contains some strings.

Definition 6 *Let Σ be an alphabet. Consider a language $L = \cup_n L_n$, where $L_n = L \cap \Sigma^n$, and let $D_L = \{n : L_n \neq \emptyset\}$. We say that L is $(1-\epsilon)$-hard for a class of circuit families $\mathcal{C}$ if D_L is an infinite set and, for any circuit family $\{C_n\}$ in $\mathcal{C}$, there exists m_0 such that for all $m \in D_L$ such that $m \geq m_0$, $Pr_{x\in\Sigma^m}[f(x) = C_m(x)] < 1-\epsilon$.*

The following theorem shows that if FWP $\notin$ io-TC^0, then BWP is hard on average for TC^0.

Theorem 7 *There exist constants $c, \delta > 0$ and $0 < \epsilon < 1$ such that for any constant $d > 0$, if* FWP$_n$ *is not computable by* TC$^0(\delta n(s(n) + cn))$ *circuits of depth at most $d + c$, then* BWP$_n$ *is $(1-\epsilon)$-hard for* TC^0 *circuits of size $s(n)$ and depth d.*

Proof. Let $\epsilon < \frac{1}{4\binom{120}{60}}$. We prove the contrapositive. Assume there is a circuit C of size $s(n)$ and depth d such that $Pr_x[\text{BWP}_n(x) = C(x)] \geq 1-\epsilon$. We first present a probabilistic algorithm for FWP$_n$.

Let the input instance for FWP$_n$ be $w_1 w_2 \ldots w_n$. Generate a sequence of $n+1$ random permutations $u_0, u_1, \ldots, u_n$ in S_5 and a random set $S \subseteq S_5$ of size 60. Let ϕ be the sequence $(u_0 \cdot w_1 \cdot u_1)(u_1^{-1} \cdot w_2 \cdot u_2)..(u_{n-1}^{-1} \cdot w_n \cdot u_n)$. Note that ϕ is a completely random sequence in S_5^n.

Let us say that ϕ is a "good" sequence if $\forall S' \subset S_5$ with $|S'| = 60,\ C(\langle \phi, S' \rangle) = \text{BWP}_n(\langle \phi, S' \rangle)$.

If we have a "good" sequence ϕ (meaning that for *every* set S' of size 60, C gives the "correct" answer BWP$_n(\phi, S')$ on input (ϕ, S')), then we can easily find the unique value r that is equal to $\Pi_{i=1}^n \phi_i$ where $\phi_i = u_{i-1} w_i u_i$, as follows:

- If $C(\phi, S) = 1$, then it must be the case that $r \in S$. Pick any element $r' \in S_5 \setminus S$ and observe that r is the only element such that $C(\phi, (S \setminus \{r\}) \cup \{r'\}) = 0$.
- If $C(\phi, S) = 0$, then it must be the case that $r \notin S$. Pick any element $r' \in S$ and observe that r is the only element such that $C(\phi, (S \setminus \{r'\}) \cup \{r\}) = 1$.

Thus the correct value r can be found by trying all such r'. Hence, if ϕ is good, we have

$$r = \Pi_{i=1}^n \phi_i = u_0 w_1 u_1 \Pi_{i=2}^n u_{i-1}^{-1} w_i u_i.$$

Produce as output the value $u_0^{-1} r u_n^{-1} = \Pi_{i=1}^n w_i = \text{FWP}_n(w)$.

Since $\epsilon < \frac{1}{4\binom{120}{60}}$, a standard averaging argument shows that at least $\frac{3}{4}$ of the sequences in S_5^n are good. Thus with probability at least $\frac{3}{4}$, the probabilistic algorithm computes FWP$_n$ correctly. The algorithm can be computed by a threshold circuit of depth $d + O(1)$ since the subroutines related to C can be invoked in parallel and moreover, the preparation of ϕ and the aggregation of results of subroutines can be done by constant-depth threshold circuits. Its size is at most $121 s(n) + O(n)$ since there are 121 calls to C. Next, we put $10^4 n$ independent copies together in parallel and output the majority vote. Let X_i be the random variable that the outcome of the ith copy is $\Pi_{i=1}^n w_i$. By Lemma 4, on every input the new circuit computes FWP$_n$ with probability at least $1 - \frac{120^{-n}}{2}$. Thus there is a random sequence that can be hardwired in to the circuit, with the property that the resulting circuit gives the correct output on *every* input (and in fact, at least half of the random sequences have this property). This yields a deterministic TC^0 circuit computing FWP$_n$ exactly which is of depth at most $d + c$ and of size no more than $(121 \cdot 10^4) n(s(n) + cn)$ for some universal constant c . Choosing $\delta \geq (121 \cdot 10^4)$ completes the proof. □

Definition 8 [AK10, Definition 5] *Let $f : \{0,1\}^* \to \{0,1\}^*$ be a function. Let $s(n), m(n) : \mathcal{N} \to \mathcal{N}$ be functions such that $m(n) < n$ for all n, and let $d \geq 1$ be an integer. We say f_n is* downward self-reducible *to $f_{m(n)}$ by a pure reduction of depth d and size $s(n)$ if there is a circuit family $\{C_n\}_{n\geq 1}$ such that for each n, C_n computes f_n, is of depth at most d and size at most $s(n)$, and consists of fan-in two* AND *and* OR *gates, unary* NOT *gates and oracle gates that compute function f on inputs of size at most $m(n)$. If f is downward self-reducible to f_{n^ϵ} for some $1 > \epsilon > 0$ we will say f is* strongly downward self-reducible.

The problem FWP is strongly downward self-reducible [AK10, Proposition 3.6].

Theorem 9 [AK10] *If there is a* $\gamma > 0$ *such that* FWP $\notin$ io-TC$^0(n^{1+\gamma})$, *then* FWP $\notin$ io-TC^0.

Proof. We briefly sketch the proof. By [AK10, Proposition 7] FWP$_n$ is strongly downward self-reducible to FWP$_{n^\epsilon}$ by a Dlogtime-uniform pure reduction of depth $O(1/\epsilon)$ and size $O(n)$. Hence, as observed in [AK10, Corollary 4.3], for any given $\gamma > 0$, an io-TC^0 circuit of polynomial size for FWP can be combined with the self-reduction for FWP (for a suitably chosen ϵ) to obtain an io-TC^0 circuit of size $n^{1+\gamma}$. □

(Theorem 9 is not stated in terms of io-TC^0 in [AK10], but the proof there shows that if there are infinitely many input lengths n where FWP has circuits of of size n^k, then there are infinitely many input lengths m where FWP has circuits of size $m^{1+\gamma}$. The strong downward self-reducibility property allows small circuits for inputs of size m to be constructed by efficiently using circuits for size $n < m$ as subcomponents.)

Since FWP is equivalent to WP via linear-size reductions on the same input length, the following corollary is its easy consequence.

Corollary 10 *If there is a* $\gamma > 0$ *such that* WP $\notin$ io-TC$^0(n^{1+\gamma})$, *then* FWP $\notin$ io-TC^0.

Combining Corollary 10 with Theorem 7 yields the average-case hardness of BWP from nearly-linear-size worst-case lower bounds for WP against TC^0 circuit families.

Corollary 11 *There exists a constant* $\epsilon > 0$ *such that if* $\exists\gamma > 0$ *such that* WP $\notin$ io-TC$^0(n^{1+\gamma})$, *then for any* k *and* d *there exists* $n_0 > 0$ *such that when* $n \geq n_0$, BWP$_n$ *is* $(1-\epsilon)$*-hard for any* TC^0 *circuit of size* n^k *and depth* d.

Define the following Boolean function WPM$_n$: $S_5^n \times S_5^{60} \to \{0,1\}$, where WPM$_n$ stands for Word Problem over Multi-set.

Definition 12 *The input to* WPM$_n$ *is a pair* $\langle w_1 w_2..w_n, v_1 v_2..v_{60}\rangle$, *where* $\forall i \in [1..n],\ w_i \in S_5$ *and* $\forall j \in [1..60],\ v_i \in S_5$. $\langle w_1 w_2..w_n, v_1 v_2..v_{60}\rangle \in$ WPM *if and only if* $\exists j \in [1..60],\ \Pi_{i=1}^n w_i = v_j$.

BWP is the restriction of WPM$_n$ to the case where all v_is are distinct. Hence, WPM inherits the average-case hardness of BWP, since any circuit that computes WPM$_n$ on a sufficiently large fraction of inputs also approximates BWP well. Formally,

Lemma 13 *There is an absolute constant* $0 < c < 1$ *such that for every* $\epsilon > 0$, *if* BWP$_n$ *is* $(1-\epsilon)$*-hard for* TC^0 *circuits of size* n^k *and depth* d, *then* WPM$_n$ *is* $(1-c\epsilon)$*-hard for* TC^0 *circuits of size* n^k *and depth* d.

Proof. Let $c = \frac{\binom{120}{60}}{(120)^{60}}$. Note that c is the probability that a sequence of 60 permutations contains no duplicates and is in sorted order. Suppose there is a circuit C with the property that $Pr_{x \in S^n \times S^{60}}[C(x) \neq \text{WPM}(x)] \leq c\epsilon$. Then the conditional probability that $C(x) \neq$ WPM(x) given that the last 60 items in x give a list in sorted order with no duplicates is at most ϵ. This yields a circuit having the same size, solving BWP with error at most ϵ, using the uniform distribution over its domain, contrary to our assumption. □

Corollary 14 *There exists a constant* $\epsilon > 0$ *such that if* $\exists\gamma > 0$ *such that* WP $\notin$ io-TC$^0(n^{1+\gamma})$, *then for any* k *and* d *there exists* $n_0 > 0$ *such that when* $n \geq n_0$, WPM$_n$ *is* $(1-\epsilon)$*-hard for* TC^0 *circuits of size* n^k *and depth* d.

Yao's XOR lemma [Yao82] is a powerful tool to boost average-case hardness. We utilize a specialized version of the XOR lemma for our purpose. Several proofs of this useful result have been published. For instance, see the text by Arora and Barak [AB09] for a proof that is based on Impagliazzo's hardcore lemma [Imp95]. For our application here, we need a version of the XOR lemma that is slightly different from the statement given by Arora and Barak. In the statement of the lemma as given by them, g is a function of the form $\{0,1\}^n \to \{0,1\}$. However, their proof works for any Boolean function g defined over any finite alphabet, because both the hardcore lemma and its application in the proof of the XOR lemma are insensitive to the encoding of the alphabet. Hence, we state the XOR Lemma in terms of functions over an alphabet set Σ. The proof presented in [AB09] yields the following version of the XOR lemma:

For any Boolean function g over some domain Σ^n, define $g^{\oplus m} : \Sigma^{nm} \to \{0,1\}$ by $g^{\oplus m}(x_1, x_2, .., x_m) = g(x_1) \oplus g(x_2) \oplus .. \oplus g(x_m)$ where $\oplus$ represents the parity function.

Lemma 15 *[Yao82] Let $\frac{1}{2} < \epsilon < 1$, $k \in \mathbb{N}$ and $\theta > 2(1-\epsilon)^k$. There is a constant $c > 1$ that depends only on $|\Sigma|$ such that if g is $(1-\epsilon)$-hard for TC^0 circuits of size s and depth d, then $g^{\oplus k}$ is $(\frac{1}{2}+\theta)$-hard for TC^0 circuits of size $\frac{\theta^2 s}{cn}$ and depth $d-1$.*

Let $\Sigma = S_5$. The following corollary is an immediate consequence of Corollary 14 and Lemma 15.

Corollary 16 *If there is a $\gamma > 0$ such that $\mathrm{WP} \notin \text{io-}\mathrm{TC}^0(n^{1+\gamma})$, then for any k, k' and d there exists $n_0 > 0$ such that when $n \geq n_0$ $(\mathrm{WPM}_n)^{\oplus n}$ is $(\frac{1}{2} + \frac{1}{n^{k'}})$-hard for TC^0 circuits of size n^k and depth d.*

Let $\mathrm{WP}^{\otimes} = \cup_{n \geq 1}\{x \mid (\mathrm{WPM}_n)^{\oplus n}(x) = 1\}$. Note that it is a language in uNC^1 and, moreover, it is decidable in linear time.

Theorem 17 *If there is a $\gamma > 0$ such that $\mathrm{WP} \notin \text{io-}\mathrm{TC}^0(n^{1+\gamma})$, then for any integer $k > 0$, $\mathrm{WP}^{\otimes}$ is $(\frac{1}{2} + \frac{1}{n^k})$-hard for TC^0.*

3.2 Worst-case to Average-case Reduction for L

Here we show a similar worst-case to average-case connection as in the previous subsection, but for the class L which contains NC^1. Just as the word problem WP is complete for NC^1, the word problem PWP for S_n is complete for L [MC87].

Definition 18 *The language* PWP *consists of all inputs $\langle w_1, w_2 \ldots w_n \rangle$, where each w_i encodes a permutation over S_n and $\Pi_{i=1}^n w_i$ is the identity permutation.*

We will use a few different encodings of permutations. Encoding 1 is where the permutation is represented simply as an ordered list of n distinct numbers between 1 and n - the interpretation of this list as a permutation is that if the k'th element in the list is j, then k maps to j in the permutation. Encoding 2 is less economical and represents a permutation as an ordered list of n ordered pairs $(i, \sigma(i))$, where i ranges from 1 to n and σ is a permutation on $[n]$. The interpretation here is that the i maps to $\sigma(i)$ in the permutation σ. Here, whether the list is ordered does not matter - all permutations of the ordered list represent the same permutation in S_n. The fact that each *pair* is ordered is of course critical.

Using the fact that Sorting is in TC^0 (e.g. see [Vol99]), we can convert from Encoding 1 to Encoding 2 or vice-versa in TC^0. The conversion from Encoding 1 to Encoding 2 is trivial - simply prefix each number in the ordered list by its index in the list. To convert from Encoding 2 to Encoding 1, sort using the first element of the ordered pair as the key, and retain only the second element in the sorted list.

For technical reasons, we will use a third even more verbose encoding - Encoding 3. In Encoding 3, a permutation σ is represented as an ordered list of n integers each of which is n bits long. The permutation represented by this list is the identity permutation if there are two elements of the list which are equal, and is otherwise the permutation σ where $\sigma(i)$ is the rank of the i'th element in the list, i.e., its index in the sorted order. Note that a permutation in Encoding 1 can be trivially converted to Encoding 3 by prefixing each element by $n - \log n$ zeroes. To convert from Encoding 3 to Encoding 1 in TC^0, first check that there are no "collisions" in the list, i.e., a pair of identical elements. If there is a collision, output the identity permutation - this can be done in AC^0. If there are no collisions, transform the ordered list to an ordered list of ordered pairs formed by pairing each element of the original list with its index in the list. Sort according to the elements of the original list, but retain only the corresponding order on the indices. If the list survives the collision check, this yields a permutation in Encoding 1.

Using the fact that the composition of two TC^0 functions is in TC^0, we get that we can convert from Encoding 2 to Encoding 3 and vice versa in TC^0.

By default, we will consider the third encoding to be in effect. If this is not the case, we will explicitly say so.

For the purpose of studying a worst-case to average-case connection for L,we need a balanced version of the language PWP.

Definition 19 *The language* BPWP *is defined as follows:*

$$\text{BPWP} = \{\langle w_1, w_2 \dots w_n, i\rangle \mid \textit{each } w_j \textit{ encodes a permutation in } S_n \textit{ w.r.t. Encoding 3 and the } i^{th} \textit{ bit of the encoding of } \textstyle\prod_{j=1}^{n} w_j \textit{ is } 1\}.$$

We assume a natural Boolean encoding of the inputs, where the only relevant inputs are of size $n^3 + 2\log n$, with n blocks of n^2 bits each representing $w_1 \dots w_n$ according to Encoding 3 and the last block representing i. We assume wlog that n is a power of 2 - BPWP remains complete for L with this restriction.

Lemma 20 *There is a family $\{C_n\}$ of randomized TC^0 circuits of polynomial size such that for each n, the output of C_n is $O(n^2/2^n)$-close in statistical distance to the uniform distribution over (σ, σ^{-1}), where σ is uniformly chosen in S_n. Moreover, when considered purely as bit strings, the first and second outputs of C_n are $O(n^2/2^n)$-close to the uniform distribution.*

Proof.

The circuits C_n are defined as follows. First, n numbers $x_1, x_2 \dots x_n$, with each $x_i, 1 \leq i \leq n$ being n bits long, are generated at random. As per Encoding 3, this n-tuple of numbers represents a permutation σ. The identity permutation is generated with probability at most $1/n! + n^2/2^n$, since the probability of a collision is at most $n^2/2^n$. Every other permutation is generated with equal probability, which is at least $(1 - n^2/2^n)1/n!$. A simple computation of the statistical distance yields that the corresponding distribution on permutations is $O(n^2/2^n)$-close to the uniform distribution on permutations σ over $[n]$.

It now remains to show how to generate σ^{-1}. Sort the x-list $x_1, x_2 \dots x_n$ - this can be done in TC^0. Then convert σ from Encoding 3 to Encoding 2 in TC^0. Then we include circuitry which *reverses* the order of each ordered pair in the list, to yield the representation of σ^{-1} according to Encoding 2. Then implement the TC^0 conversion from Encoding 2 to Encoding 1, and finally use the elements of the resulting list as ranks to select elements from the sorted x-list. We thus derive a representation of σ^{-1} according to Encoding 3 which is itself a permutation of the representation of σ according to Encoding 3. The last part of the lemma follows using this fact and the argument in the previous paragraph on the relative unlikelihood of the identity permutation being represented. □

Lemma 20 gives us the ability to generate a random permutation and its inverse efficiently. This can be used to implement a random self-reduction in TC^0 and hence derive a worst-case to average-case hardness amplification in L against TC^0.

Theorem 21 *If* L $\not\subseteq$ TC^0*, then there is a language in* L *which is* $(1 - 1/n^2)$*-hard for* TC^0.

Proof. The language for which we show a random self-reduction is BPWP. Assume that BPWP is not $(1-1/n^2)$-hard for the complexity class TC^0. We show how to solve BPWP in TC^0 based on this assumption. Since BPWP is complete for L, this implies that L $\subseteq$ TC^0.

Let $\langle w_1, w_2 \ldots w_n, i\rangle$ be an input instance for BPWP, where each w_j represents a permutation over S_n according to Encoding 3. We generate $n \log n$ randomized queries to BPWP such that for each query, the query with the last co-ordinate omitted is $1 - O(n^2/2^n)$-close to the uniform distribution over binary strings. The queries are generated in TC^0 as follows. Using Lemma 20, generate n random permutations $\sigma_1, \sigma_2, \ldots, \sigma_n$ and their inverses. We do not know how to do this exactly, but it suffices to do it approximately as guaranteed by Lemma 20. Form the permutations $s_1, s_2 \ldots s_n$, where for each $j, 1 < j \leq n$, $s_j = \sigma_{j-1}^{-1} w_j \sigma_j$, and $s_1 = w_1\sigma_1$. To form these permutations, convert to Encoding 1 and use the fact that two permutations can be multiplied in TC^0 when represented in Encoding 1. When converting back to Encoding 3, for each $j, 1 \leq j \leq n$, sort the list of numbers representing σ_j and then use the representation of the permutation in Encoding 1 as ranks to select from the sorted list. Thus for each j, the resulting permutation is exponentially close to a random permutation of the list of numbers representing σ_j. Since the σ_j are all independent, we have that $s_1 \ldots s_n$ are all independent and exponentially close to the uniform distribution as bit strings. Now form the queries $\langle s_1, s_2 \ldots s_n, k\rangle$ for each $1 \leq k \leq n \log n$.

Since BPWP is not $(1 - 1/n^2)$-hard for TC^0, the assumption on the distribution of queries implies that the TC^0 approximators for BPWP return the correct answers for all queries with probability at least $1 - (\log n)/n$, for large enough n. Using the correct answers for all queries, we can reconstruct $s_1 s_2 \ldots s_n$ in Encoding 1. Also, we know that $w_1 w_2 \ldots w_n = s_1 s_2 \ldots s_n \sigma_n^{-1}$. Thus we can reconstruct $w_1 w_2 \ldots w_n$ in Encoding 1 with another multiplication in TC^0 and then obtain its i^{th} bit correctly with high probability. Finally, by a standard amplification step followed by Adleman's trick [Adl78], this probabilistic circuit can be converted to a non-uniform one.

□

3.3 Worst-case to Average-case Reduction for GapL and GapNC^1

We first consider GapL. Let $\mathrm{Determinant}$ denote the problem of computing the integer determinant. This is a complete problem for GapL (see, e.g. [MV97]). We show that if $\mathrm{Determinant}$ cannot be computed by TC^0 circuits then $\mathrm{Determinant}$ is somewhat hard on average for TC^0 circuits. As TC^0 circuits take Boolean input, we will encode each integer entry of an $n \times n$ integer matrix in binary. In order to keep the overall size of this Boolean input bounded, we will make the simplifying assumption that each entry of an $n \times n$ integer matrix instance of $\mathrm{Determinant}$ is at most n bits long. It is not hard to see that this version of $\mathrm{Determinant}$ is also complete for GapL. Since the proof of the next theorem is similar to the standard argument for proving random self-reducibility of $\mathrm{Permanent}$ [Lip91], we omit some low-level details.

Theorem 22 *Let $\mathcal{M}_n$ denote the set of all $n \times n$ matrices where each integer entry has size at most n bits.*[1] *If there is a* TC^0 *circuit computing* $\mathrm{Determinant}$ *for at least a $1 - \frac{1}{n^5}$ fraction of inputs from $\mathcal{M}_n$ then there is a* TC^0 *circuit that computes* $\mathrm{Determinant}$ *for all inputs from $\mathcal{M}_n$.*

Proof. Let C' denote the TC^0 circuit that computes the integer determinant for $1 - \frac{1}{n^5}$ fraction of inputs from $\mathcal{M}_n$. Our goal is to construct a TC^0 circuit that computes the integer determinant for *every* input matrix $M \in \mathcal{M}_n$. For input $M \in \mathcal{M}_n$, we will describe a *nonadaptive* reduction from the problem of computing $\det(M)$ to computing $\det(M_i)$ for a sequence of random matrices $M_i \in \mathcal{M}_n, 1 \leq i \leq r$ where each M_i is nearly uniformly distributed in $\mathcal{M}_n$. To this end, pick a random matrix $A \in \mathcal{M}_n$. This requires $n^3 + n^2$ independent unbiased coin flips to

[1] It is necessary to be precise about what it means for an integer entry to have n bits. We use two's-complement notation; thus the entries come from the set $\{-2^{n-1}, \ldots, 2^{n-1} - 1\}$.

pick the n^2 random n-bit entries of A along with their signs. Now, consider the polynomial $\det(M + Ax)$. This is a degree n polynomial over $\mathbb{Z}$ in the indeterminate x. Let $S = \{1, 2, \ldots, n+1\}$ be distinct interpolating points and consider $\det(M + Ai)$ for each $i \in S$. The matrix $M + Ai$ is random. Unfortunately, it is not uniformly distributed in $\mathcal{M}_n$ (indeed, even its support is not contained in $\mathcal{M}_n$). Therefore, we cannot directly use the circuit C' to compute $\det(M + Ai)$ for all $i \in S$ and interpolate the value of $\det(M)$. We shall get around this difficulty with Chinese remaindering.

By the Hadamard bound $|\det(M)| \leq 2^{n^2} \cdot n! < 4^{n^2}$ for all $M \in \mathcal{M}_n$. We can pick n^2 distinct $O(\log n)$ bit primes $p_1, p_2, \ldots, p_{n^2}$ so that $\prod_i p_i > |\det(M)|$ for each $M \in \mathcal{M}_n$. We note that $\det(M)$ can be reconstructed from the residues $\det(M) (\mathrm{mod}\, p_i), 1 \leq i \leq n^2$ by Chinese remaindering and, moreover, this reconstruction can be done in Dlogtime-uniform TC^0 [HAB02]. Hence, it suffices to describe a TC^0 circuit family for computing $\det(M) (\mathrm{mod}\, p)$ for each $M \in \mathcal{M}_n$, where p is an $O(\log n)$ bit prime.

For a matrix $A \in \mathcal{M}_n$ picked uniformly at random, consider $\det(M + Ax) (\mathrm{mod}\, p)$. This is a degree n polynomial in x modulo p. We will compute $\det(M) (\mathrm{mod}\, p)$ by interpolation. Let $S = \{1, 2, \ldots, n+1\}$ be the distinct interpolating points in $\mathbb{F}_p$; in order to ensure that this yields more than n points in the finite field $\mathbb{F}_{p_i}$ for each of the primes p_i, we will pick $p_i > n+1$ for all i. For any fixed $s \in S$, we note that the matrix $M + As (\mathrm{mod}\, p)$ is nearly uniformly distributed over $n \times n$ matrices with $\mathbb{F}_p$ entries. To see this, consider a randomly picked integer entry A_{ij} of the matrix A, where A_{ij} is at most n bits long. Then for each $\alpha \in \{0, 1, \ldots, p-1\}$ it is easily seen that

$$\left| \frac{1}{p} - \Pr_{A_{ij}}[A_{ij} = \alpha (\mathrm{mod}\ p)] \right| \leq \frac{1}{2^n}.$$

Hence, an easy calculation shows for any specific matrix B in $\mathbb{F}_p^{n \times n}$

$$\left| \frac{1}{p^{n^2}} - \Pr_{A \in \mathcal{M}_n}[M + As = B (\mathrm{mod}\ p)] \right| \leq \frac{1}{p^{n^2} 2^{O(n)}}.$$

It follows that the statistical distance of the distribution of $M + As (\mathrm{mod}\ p)$ over $\mathbb{F}_p^{n \times n}$ to the uniform distribution is bounded by $2^{-O(n)}$.

However, as explained above, notice that we cannot directly use the circuit C' to compute $\det(M + As)$ since the entries of $M + As$ can be $O(n + \log n)$ bits long. Neither can we directly use C' to compute $\det(M + As) (\mathrm{mod}\, p)$, because the matrix $M + As (\mathrm{mod}\, p)$ has integer entries in the range $\{0, 1, \ldots, p-1\}$ and these matrices are only a $(\frac{p}{2^n})^{n^2}$ fraction of matrices in $\mathcal{M}_n$. It is possible that the output of C' is incorrect on all these matrices. We now describe the solution. Consider the onto mapping

$$f : \mathcal{M}_n \longrightarrow \mathbb{F}_p^{n \times n},$$

defined by $f(M) = M (\mathrm{mod}\, p)$. Now, consider the probability distribution on $\mathcal{M}_n$ defined by first picking a uniformly distributed random matrix $M' \in \mathbb{F}_p^{n \times n}$ and then picking a uniformly distributed random preimage matrix $M \in f^{-1}(M')$. A similar calculation as above shows that this distribution is exponentially-close to uniform. Now, we briefly sketch how to obtain a (nearly) random preimage M of M'. Let $2^{n-1} - 1 = q_p p + \ell_p$, where q_p and ℓ_p are the quotient and remainder on dividing $2^n - 1$ by the prime p. Similarly, let $-2^{n-1} = -q'_p p + \ell'_p$. For each entry $z = M'_{ij}$ of the matrix M' we uniformly pick a random positive integer r_{ij} in the range $-s'_{p,z} \leq r_{ij} \leq s_{p,z}$ (where $s'_{p,z} \in \{q'_p, q'_p - 1\}$ and $s_{p,z} \in \{q_p, q_p - 1\}$, so as to guarantee that $z + r_{i,j}$ is an n-bit integer), and set

$$M_{ij} = M'_{ij} + r_{ij} p.$$

Clearly, the matrix M thus defined is nearly-uniformly distributed in $f^{-1}(M')$ and a TC^0 circuit can nearly-randomly sample from $f^{-1}(M')$.

Now, for the random matrix $M + As (\mathrm{mod}\, p) \in \mathbb{F}_p^{n \times n}$ consider its random preimage $M_s \inf^{-1}(M + As)$, for $s \in \{1, 2, \ldots, n+1\}$. By the above argument, it follows that each M_s is statistically close to the uniform distribution on $\mathcal{M}_n$. Hence, for each $s \in S$:

$$\Pr[C'(M_s) = \det(M_s)] \geq 1 - \frac{1}{n^5} - \frac{1}{2^{O(n)}},$$

where the term $\frac{1}{2^{O(n)}}$ is subtracted in the above bound as it bounds the statistical distance of M_s's distribution from the uniform.

Hence with probability $1 - \frac{1}{n^3}$ the circuit C' correctly computes $\det(M_s)$ for all $s \in S$. Now, applying the fact that polynomial interpolation is TC^0 computable [HAB02], a TC^0 circuit can recover $\det(M) (\mathrm{mod}\, p)$, given $\det(M_s) (\mathrm{mod}\, p)$ for all $s \in S$.

Putting it together, for each prime p_i we have a randomized TC^0 circuit that computes $\det(M) (\mathrm{mod}\, p_i)$ with probability $1 - \frac{1}{n^3}$. Finally, applying Chinese remaindering which is TC^0 computable [HAB02], we obtain a randomized TC^0 circuit that computes $\det(M)$ with probability $1 - \frac{1}{n}$. As before, the random bits can be fixed after amplifying the success probability using Adleman's trick [Adl78]. □

We now briefly discuss a similar worst-case to average-case reduction for GapNC^1. The problem of computing the $(1,1)^{th}$ entry of the product of n 3×3 integer matrices is GapNC^1 complete [CMTV98]. We show that if this problem cannot be computed by TC^0 circuits then it is somewhat hard on average for TC^0 circuits. As before, since we consider TC^0 circuits which take Boolean inputs, we consider inputs $(M_1, M_2, \cdots, M_n)$ in a smaller set $\mathcal{I}_n$ such that each M_i is a 3×3 matrix with integer entries that are at most n bits long. This restricted problem is also easily seen to be GapNC^1 complete. In order to show the worst-case to average-case reduction we pick a uniform random instance $(A_1, A_2, \cdots, A_n) \in \mathcal{I}_n$ and consider the instance $(M_1 + A_1 x, M_2 + A_2 x, \cdots, M_n + A_n x)$ for indeterminate x. Notice that the $(1,1)^{th}$ entry of the matrix $\prod_{i=1}^{n} (M_i + A_i x)$ is a degree n polynomial in x. Now, exactly along the same lines as the proof of Theorem 22 we can show the following.

Theorem 23 *Let $\mathcal{I}_n$ denote all iterated matrix multiplication instances $M_1, M_2, \cdots, M_n$, consisting of 3×3 integer matrices M_i whose entries are at most n bits long. If there is a TC^0 circuit computing $\prod_{i=1}^{n} M_i$ for at least $1 - \frac{1}{n^5}$ inputs $M_1, M_2, \cdots, M_n$ in $\mathcal{I}_n$ then there is a TC^0 circuit that computes $\prod_{i=1}^{n} M_i$ for all inputs $M_1, M_2, \cdots, M_n$ in $\mathcal{I}_n$.*

4 Uniform derandomization

The Nisan-Wigderson generator is the canonical method to prove the existence of pseudo-random generators based on hard functions. It relies on the following definition of combinatorial designs.

Definition 24 (Combinatorial Designs) *Fix a universe of size u. An (m, l)-design of size n on $[u]$ is a list of subsets $S_1, S_2, ..., S_n$ satisfying:*

1. $\forall i \in [1..n],\ |S_i| = m$;
2. $\forall i \neq j \in [1..n],\ |S_i \cap S_j| \leq l$.

Nisan and Wigderson [NW94] invented a general approach to construct combinatorial designs for various ranges of parameters. The proof given by Nisan and Wigderson gives designs where $l = \log n$, and most applications have used that value of l. For our application, l can be considerably smaller, and furthermore, we need the S_i's to be very efficiently computable. For completeness, we present the details here. (Other variants of the Nisan-Wigderson construction have been developed for different settings; we refer the reader to one such construction by Viola [Vio05], as well as to a survey of related work [Vio05, Remark 5.3].)

Lemma 25 *[vL99] For $l > 0$, the polynomial $x^{2 \cdot 3^l} + x^{3^l} + 1$ is irreducible over $\mathbb{F}_2[x]$.*

Lemma 26 *[NW94] For any integer n, any α such that $\log \log n / \log n < \alpha < 1$, let $b = \lceil \alpha^{-1} \rceil$ and $m = \lceil n^\alpha \rceil$, there is a (m, b)-design with $u = O(m^6)$. Furthermore, each S_i can be computed within $O(bm^2)$ time.*

Proof. Fix $q = 2^{2 \cdot 3^l}$ for some l such that $m \leq q \leq m^3$. Let the universe for the combinatorial design construction be $\mathbb{F}_q \times \mathbb{F}_q$. Let $p_1.p_2, \ldots, p_n$ be the lexicographically first n univariate polynomials of degree at most b over $\mathbb{F}_q$, and let $S_i = \{(a, p_i(a)) \mid a \in \mathbb{F}_q\}$ be the graph of the polynomial p_i. Since $q^b \geq (n^\alpha)^b \geq n$, there are at least n such distinct polynomials p_i and hence such sets S_i. No two polynomials share more than b points which implies the second condition of Definition 24. The first condition holds because we could simply drop elements from any S_i without increasing the size of intersections.

The arithmetic operations in $\mathbb{F}_q$ are performed in $\log^{O(1)} q$ time because of the explicitness of the irreducible polynomial given by Lemma 25. It is evident that for any $i \in [n]$, we can enumerate all elements of S_i in time $O(m \cdot b(\log^{O(1)} q)) = O(bm^2)$. □

Lemma 27 *For any constant $\alpha > 0$ and for any large enough integer n, if g is $(\frac{1}{2} + \frac{1}{n^2})$-hard for TC^0 circuits of size n^2 and depth $d + 2$, then any probabilistic TC^0 circuit C of size n and depth d can be simulated by another probabilistic TC^0 circuit of size $O(n^{1+\alpha})$ and depth $d + 1$ which is given oracle access to $g_{\lceil n^\alpha \rceil}$ and uses at most $O(n^{6\alpha})$ many random bits.*

Proof. This is a direct consequence of Lemma 26; we adapt the traditional Nisan-Wigderson argument to the setting of TC^0 circuits. Let n and α be given, with $0 < \alpha < 1$. Let $S_1, \ldots, S_n$ be the (m, b)-design from Lemma 26, where $m = \lceil n^\alpha \rceil$, $b = \lceil \alpha^{-1} \rceil$, and each $S_i \subset [u]$, with $u = O(m^6)$. We are given $g : \Sigma^m \to \{0,1\}$; define $h^g : \Sigma^u \to \{0,1\}^n$ by $h^g(x) = g(x|_{S_1})g(x|_{S_2})..g(x|_{S_n})$, where $x|_{S_i}$ is the sub-sequence restricted to the coordinates specified by S_i.

The new circuit samples randomness uniformly from Σ^u and feeds C with pseudo-random bits generated by h^g instead of purely random bits. It only has one more extra layer of oracle gates and its size is bounded by $O(n + n \cdot n^\alpha) = O(n^{1+\alpha})$. What is left is to prove the following claim.

Claim 28 *For any constant $\epsilon > 0$, $|Pr_{x \in \{0,1\}^n}[C(x) = 1] - Pr_{y \in \Sigma^u}[C(h^g(y)) = 1]| < \epsilon$.*

Proof. Suppose there exists ϵ such that $|Pr_{x \in \{0,1\}^n}[C(x) = 1] - Pr_{y \in \Sigma^u}[C(h^g(y)) = 1]| \geq \epsilon$. We will seek a contradiction to the hardness of g via a hybrid argument.

Sample z uniformly from Σ^u and r uniformly from $\{0,1\}^n$. Create a sequence of $n + 1$ distributions H_i on $\{0,1\}^n$ where

- $H_0 = r$;
- $H_n = h^g(z)$;
- $\forall 1 \leq i \leq n - 1,\ H_i = h^g(z)_1 h^g(z)_2 \ldots h^g(z)_i r_{i+1} \ldots r_n$.

By our assumption, $|\Sigma_{j=1}^n (Pr_{x \sim H_{j-1}}[C(x) = 1] - Pr_{x \sim H_j}[C(x) = 1])| \geq \epsilon$. Therefore, $\exists i \in [n]$ such that $|Pr_{x \sim H_{i-1}}[C(x) = 1] - Pr_{x \sim H_i}[C(x) = 1]| \geq \frac{\epsilon}{n}$.

Assume $Pr_{x \sim H_i}[C(x) = 1] - Pr_{x \sim H_{i-1}}[C(x) = 1] \geq \frac{\epsilon}{n}$, otherwise add a NOT gate at the top of C, and treat the new circuit as C instead.

Consider the following probabilistic TC^0 circuit C' for the function g. On input x, the circuit C' samples z uniformly from Σ^u and r uniformly from $\{0,1\}^n$ and replaces the substring $z|_{S_i}$ of z (i.e. the substring whose coordinates are indexed by S_i) with the input string x. Then the circuit C' samples a random bit $b \in \{0,1\}$. If

$C(h^g(z)_1 \dots h^g(z)_{i-1} b r_{i+1} \dots r_n) = 1$, C' outputs b, otherwise, it outputs $1-b$. For an input string $x \in \Sigma^m$ picked uniformly at random we now lower bound the probability that C' computes the function g.

Let y denote the random string $h^g(z)_1 \dots h^g(z)_{i-1} b r_{i+1} \dots r_n$ which is the distribution H_{i-1}. Further, let $p_{i-1} = \Pr_{y \sim H_{i-1}}[C(y) = 1]$ and $p_i = \Pr_{w \sim H_i}[C(w) = 1]$. In the following expressions all probabilities are over uniformly picked strings $x \in \Sigma^m$, $z \in \Sigma^u$ and $r \in \{0,1\}^n$.

$$\begin{aligned}
Pr[C'(x) = g(x)] &= \\
&= Pr[C'(x) = b \ \wedge \ b = g(x)] + Pr[C'(x) \neq b \ \wedge \ b \neq g(x)] \\
&= Pr[C(y) = 1 \ \wedge \ b = g(x)] + Pr[C(y) = 0 \ \wedge \ b \neq g(x)] \\
&= \frac{1}{2} p_i + \frac{1}{2} Pr[C(y) = 0 \mid b \neq g(x)] \\
&= \frac{1}{2} p_i + \frac{1}{2} - \frac{1}{2}\alpha
\end{aligned}$$

where $\alpha = Pr[C(y) = 1 \mid b \neq g(x)]$. Observe that

$$\begin{aligned}
p_{i-1} = \Pr[C(y) = 1] &= \Pr[C(y) = 1 \wedge g(x) = b] + \Pr[C(y) = 1 \wedge g(x) \neq b] \\
&= \frac{1}{2} p_i + \frac{1}{2}\alpha.
\end{aligned}$$

Substituting above for α above we get

$$Pr_{x,z,r}[C'(x) = g(x)] = \frac{1}{2} + p_i - p_{i-1} \geq \frac{1}{2} + \frac{\epsilon}{n}.$$

By an averaging argument we can fix z, r and b and hardwire into C' to get a new circuit C'' such that

$$Pr_{x \sim \Sigma^m}[C''(x) = g(x)] \geq \frac{1}{2} + \frac{\epsilon}{n}.$$

Note that in this case $\forall 1 \leq k \leq i-1, \ h^g(z)_k$ is function on input $x|_{S_k \cap S_i}$. Since $\forall k \neq i, \ |S_i \cap S_k| \leq b$, we only need a TC^0 circuit of size at most $2^{O(b)}$ and of depth at most 2 to compute each $h^g(z)_k$. In conclusion, we obtain a TC^0 circuit C''' of size at most $(2^{O(b)}+1)n$ and of depth at most $d+2$ such that $Pr_{x \in \Sigma^m}[C'''(x) = g(x)] \geq \frac{1}{2} + \frac{\epsilon}{n} \geq \frac{1}{2} + \frac{1}{n^2}$ when n is large enough, which is a contradiction. □

□

The simulation in Lemma 27 is quite uniform, thus, plugging in appropriate segments of $\mathrm{WP}^{\otimes}$ as our candidates for the hard function g, we derive our first main result.

Theorem 29 *If* WP *is not infinitely often computed by* $\mathrm{TC}^0(n^{1+\gamma})$ *circuit families for some constant* $\gamma > 0$*, then any language accepted by polynomial-size probabilistic uniform* TC^0 *circuit family is in* $\mathsf{uTC}^0(\mathsf{SUBEXP})$.

Proof. Fix any small constant $\delta > 0$. Let L be a language accepted by some probabilistic uniform TC^0 circuit family of size at most n^k and of depth at most d for some constants k, d.

Choose m such that $n^{\frac{\delta}{12}} \leq m \leq n^{\frac{\delta}{6}}$, and let α be such that $m = n^{\alpha}$. By Theorem 17, when m is large enough, $\mathrm{WP}^{\otimes}_m$ is $(\frac{1}{2} + \frac{1}{n^{2k}})$-hard for TC^0 circuits of size n^{2k} and depth $d+c$, where c is any constant. Hence, as a consequence of Lemma 27, we obtain a probabilistic oracle TC^0 circuit for L_n of depth $d+1$. Since the computation only needs $O(m^6)$ random bits, it can be turned into a deterministic oracle TC^0 circuit of depth $d+2$ and of size at most $O(n^{2k}) \cdot 2^{O(m^6)} \leq 2^{O(n^{\delta})}$ (when n is large enough), where we evaluate the previous circuit

on every possible random string and add an extra MAJORITY gate at the top. The oracle gates all have fan-in $m \leq n^{\delta/6}$, and thus can be replaced by DNF circuits of size $2^{O(n^\delta)}$, yielding a deterministic TC^0 circuit of size $2^{O(n^\delta)}$ and depth $d+3$.

We need to show that this construction is uniform, so that the direct connection language can be recognized in time $O(n^\delta)$. The analysis consists of three parts.

- The connectivity between the top gate and the output gate of individual copies is obviously computable in time $m^6 \leq n^\delta$.
- The connectivity inside individual copies is DLOGTIME-uniform, hence, n^δ-uniform.
- By Lemma 26 each S_i is computable in time $O(bm^2)$ which is $O(m^2)$ since b is a constant only depending on δ. Moreover, notice that $\mathrm{WP}^\otimes$ is a linear-time decidable language. Therefore, the DNF expression corresponding to each oracle gate can be computed within time $O(m^2) \leq n^\delta$.

In conclusion, the above construction produces a uniform TC^0 circuit of size $2^{O(n^\delta)}$. Since δ is arbitrarily chosen, our statement holds. □

5 Consequences of pathetic arithmetic circuit lower bounds

In this section we show that a pathetic lower bound assumption for *arithmetic circuits* yields a uniform derandomization of a special case of polynomial identity testing (introduced and studied by Dvir *et al.* [DSY09]).

The explicit polynomial that we consider is $\{\mathrm{IMM}_n\}_{n>0}$, where IMM_n is the $(1,1)^{th}$ entry of the product of n 3×3 matrices whose entries are all distinct indeterminates. Notice that IMM_n is a degree n multilinear polynomial in $9n$ indeterminates, and IMM_n can be considered as a polynomial over any field $\mathbb{F}$.

Arithmetic circuits computing a polynomial in the ring $\mathbb{F}[x_1, x_2, \ldots, x_n]$ are directed acyclic graphs with the indegree zero nodes (the input nodes) labeled by either a variable x_i or a scalar constant. Each internal node is either a $+$ gate or a $\times$ gate, and the circuit *computes* the polynomial that is naturally computed at the output gate. The circuit is a *formula* if the fanout of each gate is 1.

Before going further, we pause to clarify a point of possible confusion. There is another way that an arithmetic circuit C can be said to compute a given polynomial $f(x_1, x_2, \ldots, x_n)$ over a field $\mathbb{F}$; even if C does not compute f in the sense described in the preceding paragraph, it can still be the case that for all scalars $a_i \in \mathbb{F}$ we have $f(a_1, \ldots, a_n) = C(a_1, \ldots, a_n)$. In this case, we say that C *functionally* computes f over $\mathbb{F}$. If the field size is larger than the syntactic degree of circuit C and the degree of f, then the two notions coincide. Assuming that f is not *functionally* computed by a class of circuits is a *stronger* assumption than assuming that f is not computed by a class of circuits (in the usual sense). In our work in this paper, we use the weaker intractability assumption.

An *oracle* arithmetic circuit is one that has *oracle* gates: For a given sequence of polynomials $A = \{A_n\}$ as oracle, an oracle gate of fan-in n in the circuit evaluates the n-variate polynomial A_n on the values carried by its n input wires. An oracle arithmetic circuit is called *pure* (following [AK10]) if all non-oracle gates are of bounded fan-in. (Note that this use of the term "pure" is unrelated to the "pure" arithmetic circuits defined by Nisan and Wigderson [NW97].)

The class of polynomials computed by polynomial-size arithmetic formulas is known as arithmetic NC^1. By [BOC92] the polynomial IMM_n is complete for this class. Whether IMM_n has polynomial size *constant-depth* arithmetic circuits is a long-standing open problem in the area of arithmetic circuits [NW97]. In this context, the known lower bound result is that IMM_n requires exponential size multilinear depth-3 circuits [NW97].

Very little is known about lower bounds for general constant-depth arithmetic circuits, compared to what is known about constant-depth Boolean circuits. Exponential lower bounds for depth-3 arithmetic circuits over finite fields were shown in [GK98] and [GR00]. On the other hand, for depth-3 arithmetic circuits over fields of

characteristic zero only quadratic lower bounds are known [SW01]. However, it is shown in [RY09] that the determinant and the permanent require exponential size *multilinear* constant-depth arithmetic circuits. More details on the current status of arithmetic circuit lower bounds can be found in Raz's paper [Raz10, Section 1.3].

Definition 30 *We say that a sequence of polynomials $\{p_n\}_{n>0}$ in $\mathbb{F}[x_1, x_2, \ldots, x_n]$ is $(s(n), m(n), d)$-downward self-reducible if there is a pure oracle arithmetic circuit C_n of depth $O(d)$ and with $O(s(n))$ many oracle gates that computes the polynomial p_n using oracle gates only for $p_{m'}$, for $m' \leq m(n)$.*

Analogous to [AK10, Proposition 7], we can easily observe the following. It is a direct divide and conquer argument using the iterated product structure.

Lemma 31 *For each $1 > \epsilon > 0$ the polynomial sequence $\{IMM_n\}$ is $(n^{1-\epsilon}, n^{\epsilon}, 1/\epsilon)$-downward self-reducible.*

An easy argument, analogous to the proof sketch given for Theorem 9, shows that Lemma 31 allows for the amplification of weak lower bounds for $\{\text{IMM}_n\}$ against arithmetic circuits of constant depth.

Theorem 32 *Suppose there is a constant $\delta > 0$ such that for all d and every n, the polynomial sequence $\{IMM_n\}$ requires depth-d arithmetic circuits of size at least $n^{1+\delta}$. Then, for any constant depth d the sequence $\{IMM_n\}$ is not computable by depth-d arithmetic circuits of size n^k for any constant $k > 0$.*

Our goal is to apply Theorem 32 to derandomize a special case of polynomial identity testing (first studied in [DSY09]). To this end we restate a result of Dvir et. al [DSY09].

Theorem 33 (Theorem 4 in [DSY09]) *Let n, s, r, m, t, d be integers such that $s \geq n$. Let $\mathbb{F}$ be a field which has at least $2mt$ elements. Let $P(x, y) \in \mathbb{F}[x_1, \ldots, x_n, y]$ be a non-zero polynomial with $\deg(P) \leq t$ and $\deg_y(P) \leq r$ such that P has an arithmetic circuit of size s and depth d over $\mathbb{F}$. Let $f(x) \in \mathbb{F}[x_1, \ldots, x_n]$ be a polynomial with $\deg(f) = m$ such that $P(x, f(x)) \equiv 0$. Then $f(x)$ can be computed by a circuit of size $s' = poly(s, m^r)$ and depth $d' = d + O(1)$ over $\mathbb{F}$.*

Let the underlying field $\mathbb{F}$ be large enough ($\mathbb{Q}$, for instance). The following lemma is a variant of Lemma 4.1 in [DSY09]. For completeness, we provide its proof here.

Lemma 34 (Variant of Lemma 4.1 in [DSY09]) *Let n, r, s be integers and let $f \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ be a nonzero polynomial with individual degrees at most r that is computed by an arithmetic circuit of size $s \geq n$ and depth d. Let $m = \lceil n^{\alpha} \rceil$ where $\alpha > 0$ is an arbitrary constant. Let $S_1, S_2, \ldots, S_n$ be the sets of the (m, b)-design constructed in Lemma 26 where $b = \lceil \frac{1}{\alpha} \rceil$. Let $p \in \mathbb{F}[z_1, \ldots, z_m]$ be a multilinear polynomial with the property that*

$$F(y) = F(y_1, y_2, \ldots, y_u) \triangleq f(p(y|_{S_1}), \ldots, p(y|_{S_n})) \equiv 0 \tag{1}$$

Then there exists absolute constants a and k such that $p(z)$ is computable by an arithmetic circuit over $\mathbb{F}$ with size bounded by $O((sm^r)^a)$ and having depth $d + k$.

Proof. Consider the following set of hybrid polynomials:

$$\begin{aligned} F_0(x, y) &= f(x_1, x_2, \ldots, x_n) \\ F_1(x, y) &= f(p(y|_{S_1}), x_2, \ldots, x_n) \\ &\vdots \\ F_n(x, y) &= f(p(y|_{S_1}), \ldots, p(y|_{S_n})) \end{aligned}$$

The assumption implies that $F_0 \not\equiv 0$ while $F_n \equiv 0$. Hence, there exists $0 \leq i < n$ such that $F_i \not\equiv 0$ and $F_{i+1} \equiv 0$. Notice that F_i is a nonzero polynomial in the variables $\{x_j \mid i + 1 \leq j \leq n\}$ and the variables $\{y_j \mid j \in S_1 \cup S_2 \cup \cdots \cup S_i\}$.

We recall the well-known Schwartz-Zippel lemma.

Lemma 35 (Schwartz-Zippel) [Sch80, Zip79] *Let $\mathbb{F}$ be a field and let $f \in \mathbb{F}[x_1, ..., x_n]$ be a non-zero polynomial with total degree at most r. Then for any finite subset $S \subset \mathbb{F}$ we have*

$$|\{c \in S^n \ : \ f(c) = 0\}| \leq r \cdot |S|^{n-1} \tag{2}$$

Since $\deg(F_i) \leq nrm$, then if we assume that $\mathbb{F}$ has size more than nrm, Lemma 35 assures that we can assign values from the field $\mathbb{F}$ to the variables $\{x_j \mid i+1 \leq j \leq n\}$ and the variables $\{y_j \mid j \notin S_{i+1}\}$ so that F_i remains a nonzero polynomial in the remaining variables. More precisely, fixing these variables to scalar values yields a polynomial $\tilde{f}$ with the property that

$$\begin{aligned} \tilde{f}(q_1(y|_{S_1 \cap S_{i+1}}), \ldots, q_i(y|_{S_i \cap S_{i+1}}), x_{i+1}) &\not\equiv 0 \\ \tilde{f}(q_1(y|_{S_1 \cap S_{i+1}}), \ldots, q_i(y|_{S_i \cap S_{i+1}}), p(y|_{S_{i+1}})) &\equiv 0 \end{aligned}$$

where $q_j(y|_{S_j \cap S_{i+1}})$ is the polynomial obtained from $p_j(y|_{S_j})$ after fixing the variables in $S_j \setminus S_{i+1}$.

Rename the variables $\{y_j \mid j \in S_{i+1}\}$ with $\{z_j \mid 1 \leq j \leq m\}$ and replace x_{i+1} by w. We obtain a polynomial g with the property that

$$\begin{aligned} g(z_1, \ldots, z_m, w) &\not\equiv 0 \\ g(z_1, \ldots, z_m, p(z_1, \ldots, z_m)) &\equiv 0 \end{aligned}$$

In order to apply Theorem 33, the only thing that remains is to calculate the circuit complexity of g. $\forall j \neq i+1$, $|S_j \cap S_{i+1}| \leq b$ which is a constant. Notice that, for each $j \leq i$, the polynomial $q_j(y|_{S_j \cap S_{i+1}})$ depends only on a constant (bounded by b) number of variables and is of constant degree since p is multilinear. Hence each polynomial $q_j(y|_{S_j \cap S_{i+1}})$ is a sum of at most 2^b many multilinear monomials and can be computed by a 2^b size arithmetic circuit of depth 2. Therefore, under the assumption that f has a circuit of size s and depth d, g is computable by a circuit of size $s + O(n)$ and depth $d+2$ which is a composition of the aforementioned circuits. It is important to note that $\deg_w(g) = \deg_{x_{i+1}}(f) \leq r$.

Now we can use Theorem 33 to obtain that $p(z)$ has a circuit of depth $d+k$ and size at most $(sm^r)^a$, for some constant a. This concludes the proof. □

At this point we describe our deterministic black-box identity testing algorithm for constant-depth arithmetic circuits of polynomial size and bounded individual degree. Let n, m, u, α be the parameters as in Lemma 26. Given such a circuit C over variables $\{x_i \mid i \in [n]\}$ of size $s = n^t$, depth d and individual degree r, we simply replace x_i with IMM$(y|S_i)$ where y is a new set of variables $\{y_j \mid j \in [u]\}$. Let $\tilde{C}[y_1, \ldots, y_u]$ denote the polynomial computed by the new circuit.

Notice that the total degree of $\tilde{C}$ is bounded by u^c where c is a constant depending on the combinatorial design and r. Let $R \subseteq \mathbb{F}$ be any set of $u^c + 1$ distinct points. Then by Lemma 35 the polynomial computed by $\tilde{C}$ is identically zero if and only if $\tilde{C}(a_1, a_2, \ldots, a_u) = 0$ for all $(a_1, a_2, \ldots, a_u) \in R^u$.

This gives us the claimed algorithm. Its running time is bounded by $O((u^c+1)^u) = O(2^{n^{7\alpha}})$. Since α can be chosen to be arbitrarily small, we have shown that this identity testing problem is in deterministic sub-exponential time. The correctness of the algorithm follows from the next lemma.

Lemma 36 *If for every constant $d' > 0$, the polynomial sequence $\{IMM_n\}$ is not computable by depth-d' arithmetic circuits of size n^ℓ for any $\ell > 0$, then $C[x_1, \ldots, x_n] \equiv 0$ if and only if $\tilde{C}[y_1, \ldots, y_u] \equiv 0$.*

Proof. The only-if part is easy to see. Let us focus on the if part. Suppose it is not the case, which means that $\tilde{C}[y_1, \ldots, y_u] \equiv 0$ but $C[x_1, \ldots, x_n] \not\equiv 0$. Then let $C[x_1, \ldots, x_n]$ play the role of $f[x_1, \ldots, x_n]$ in Lemma 34 and let IMM$[z_1, \ldots, z_m]$ take the place of $p[z_1, \ldots, z_m]$. Therefore, IMM$[z_1, \ldots, z_m]$ is computable by a circuit of depth $d+k$ and size at most $(n^t m^r)^a = m^{O(1)}$, where k is the constant in Lemma 34 and n^t is the size of C. This contradicts the hardness assumption about $\{\text{IMM}_n\}$. □

Putting it together, we get the following result.

Theorem 37 *If there exists $\delta > 0$ such that for any constant $e > 0$, IMM requires depth-e arithmetic circuits of size at least $n^{1+\delta}$, then the black-box identity testing problem for constant-depth arithmetic circuits of polynomial size and bounded individual degree is in deterministic sub-exponential time.*

Next, we notice that the above upper bound can be sharpened considerably. The algorithm simply takes the OR over subexponentially-many evaluations of an arithmetic circuit; if any of the evaluations does not evaluate to zero, then we know that the expressions are not equivalent; otherwise they are. Note that evaluating an arithmetic circuit can be accomplished in logspace. (When evaluating a circuit over $\mathbb{Q}$, this is shown in [HAB02, Corollary 6.8]; the argument for other fields is similar, using standard results about the complexity of field arithmetic.) Note also that every language computable in logspace has AC^0 circuits of subexponential size. (This appears to have been observed first by Gutfreund and Viola [GV04]; see also [AHM+08] for a proof.) This yields the following uniform derandomization result.

Theorem 38 *If there are no constant-depth arithmetic circuits of size $n^{1+\epsilon}$ for the polynomial sequence $\{IMM_n\}$, then for every constant d, black-box identity testing for depth-d arithmetic circuits with bounded individual degree can be performed by a uniform family of constant-depth AC^0 circuits of subexponential size.*

We call attention to an interesting difference between Theorems 29 and 38. In Theorem 38, in order to solve the identity testing problem with uniform AC^0 circuits of size 2^{n^ϵ} for smaller and smaller ϵ, the depth of the AC^0 circuits increases as ϵ decreases. In contrast, in order to obtain a deterministic threshold circuit of size 2^{n^ϵ} to simulate a given probabilistic TC^0 algorithm, the argument that we present in the proof of Theorem 29 gives a circuit whose depth is not affected by the choice of ϵ. We do not know if a similar improvement of Theorem 38 is possible, but we observe here that the depth need not depend on ϵ if we use threshold circuits for the identity test.

Theorem 39 *If there are no constant-depth arithmetic circuits of size $n^{1+\epsilon}$ for the polynomial sequence $\{IMM_n\}$, then there is a constant c such that, for every constant d and every $\gamma > 0$, black-box identity testing for depth-d arithmetic circuits with bounded individual degree can be performed by a uniform family of depth $d + c$ threshold circuits of size 2^{n^γ}.*

Proof. We provide only a sketch. Choose $\alpha < \gamma/14$, where α is the constant from the discussion in the paragraph before Lemma 36. Thus, our identity testing algorithm will evaluate a depth d arithmetic circuit $C(x_1, \ldots, x_n)$ at fewer than $2^{n^{\gamma/2}}$ points $\vec{v} = (v_1, \ldots, v_n)$, where each v_i is obtained by computing an instance of IMM_{n^α} consisting of n^α 3-by-3 matrices, whose entries without loss of generality have representations having length at most n^α. Thus these instances of IMM have DNF representations of size $2^{O(n^{2\alpha})}$. These DNF representations are uniform, since the direct connection language can be evaluated by computing, for a given input assignment to IMM_{n^α}, the product of the matrices represented by that assignment, which takes time at most $(n^\alpha)^3 < \log(2^{n^{\gamma/2}})$. Evaluating the circuit C on $\vec{v}$ can be done in uniform TC^0 [AAD00, HAB02]. □

Acknowledgments

We thank Luke Friedman for many helpful discussions, and we thank Lance Fortnow for some useful suggestions. We are grateful to the referees for their helpful comments and corrections.

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