Notes for Lecture 4
Malka Rosenthal

Proof of Statement C of Switching Lemma, con't

If $|dom(\pi_1)| \geq s$, let S be the first s variables in $dom(\pi_1)$ and let $\sigma = \tilde{\pi}_1|_S$.Otherwise,
Note: There exists some i with $\rho\pi_1(D_i) \neq 1$ since otherwise $f|_{\rho\pi_1} \equiv 1$. This is impossible as $f|\rho \not\equiv 1$(as earlier in proof) and π_1 only sets fewer than s literals in C_j, a conjunct with at least $s+1$ variables. Let

$$i_2 = \min\{i : \rho\pi_1(D_i) \neq 1\}$$

Let

$$S_2 = (D_{i_2} \setminus dom(\rho\pi_1)) \cap dom(\pi).$$

Let $\pi_2 = \pi|_{S_2}$.
Define $\tilde{\pi}_2$ as follows:

$$\tilde{\pi}_2(i) = \begin{cases} * & i \notin S_2 \\ 1 & \text{``}\overline{x_i}\text{''} \in D_{i_2} \\ 0 & \text{``}x_i\text{''} \in D_{i_2} \end{cases}$$

Thus:

- $dom(\pi_2) = dom(\tilde{\pi}_2)$.
- $\pi_2 \neq \tilde{\pi}_2$, as, for example, $\rho\pi_2(D_{i_2}) = 1$ and $\rho\tilde{\pi}_2(D_{i_2}) \neq 1$.
- $\rho\pi_1(D_{i_2}) = *$. (It is $\neq 0$ as π_1 can be extended to π which makes D_{i_2} true.)
- $\forall\ l < i_2\ \rho\pi_1(D_l) = 1$. (By def'n of i_2)
- For any setting π' of the literals in $dom(\pi) \setminus dom(\pi_1\pi_2)$, we have

$$\begin{cases} \rho\pi_1\tilde{\pi}_2\pi'(D_{i_2}) \in \{0, *\} \\ \forall l < i_2\ \rho\pi_1\tilde{\pi}_2\pi'(D_l) = 1. \end{cases}$$

If $|dom(\pi_1\pi_2)| \geq s$,let S be the first s variables in $dom(\pi_1\pi_2)$ and let $\sigma = \tilde{\pi}_2|_S$. Otherwise, $f|_{\rho\pi_1\pi_2}$ is still non-trivial and this process can be repeated to define $\pi_3, \tilde{\pi}_3, \ldots, \pi_k, \tilde{\pi}_k$ where $k \leq s$. (Each time π_i and $\tilde{\pi}_i$ is defined, at least one variable is set.)
Let

$$\rho' = \rho\tilde{\pi}_1\tilde{\pi}_2 \ldots \tilde{\pi}_{k-1}\sigma.$$

Note that $\rho' \in R^{l-s}$.
Goal:Show that $K(\rho|\rho', f, l, s)$ is small.
Define $\gamma_j \in \{0, 1, *\}^t$ for $1 \leq j \leq s$ as follows:

- For $1 \leq j \leq k-1$, γ_j will describe how (in which places) π_j and $\tilde{\pi}_j$ differ. Let D_{i_j} be a disjunction of literals on the variables $\{x_{j_1} \vee \ldots \vee x_{j_r}\}$. Let the l^{th} bit of γ_j,

$$(\gamma_j)_l = \begin{cases} * & x_{j_l} \notin dom(\pi_j) \text{ or } l > r \\ 0 & \pi_j(x_{j_l}) = \tilde{\pi}_j(x_{j_l}) \\ 1 & \pi_j(x_{j_l}) \neq \tilde{\pi}_j(x_{j_l}) \end{cases}$$

- Let γ_k be as follows: Let D_{i_k} be a disjunction of literals on the variables $\{x_{k_1} \vee \ldots \vee x_{k_{r'}}\}$ and let the l^{th} bit of γ_k,

$$(\gamma_k)_l = \begin{cases} * & x_{k_l} \notin dom(\sigma) \text{ or } l > r' \\ 0 & \text{otherwise} \end{cases}$$

- For $k \leq j \leq s$, let $\gamma_j = \{*\}^t$.

Let $\gamma = \gamma_1\gamma_2\ldots\gamma_s$ (concatenate the strings together).
Note that $|\gamma| = st$.
Note: γ contains exactly s symbols which are not equal to $*$ as

$$|dom(\pi_1 \ldots \pi_{k-1}\sigma)| = s = |dom(\tilde{\pi}_1 \ldots \tilde{\pi}_{k-1}\sigma)|.$$

Thus γ is of the form

$$*^{n_0} b_1 *^{n_1} \ldots b_s *^{n_s}$$

where $b_i \in \{0,1\}$ for $0 \leq i \leq s$ and $0 \leq n_i \leq 2t$ for $0 \leq i \leq s-1$. This is because each γ_j must contain at least one bit $\in \{0,1\}$until there have been s bits $\neq \{*\}$.

Therefore, to describe γ given s and t, we can use a string of the form $\overline{z}y_{n_i}y_{b_i}$ with z giving instructions to interpret the next $s \log 2t = | y_{n_i} |$ bits as values of $n_1, \ldots, n_s$ (as $n_i \leq 2t$ for $1 \leq i \leq s-1$) and to interpret y_{b_i} with $| y_{b_i} | = s$ as the s b_i's.
We have shown that

$$K(\gamma|s,t) \leq s\log 2t + s + c_2 \tag{1}$$

Claim: $K(\rho|f,l,s) \leq \log \binom{n}{l-s} + n - l + s\log 8t + c$.
Proof: Given f, l, s, we can build ρ with a description of the form $\overline{z}y_{\rho'}y_\gamma$ where $y_{\rho'}$ is a string of length $\log\binom{n}{l-s} + n - l + s + c_1$ and y_γ is a string of length $s\log 2t + s + c_2$.
Building such a $y_{\rho'}$ is possible as $\rho' \in R^{l-s}$ and building such a $y_\gamma$is possible by (1) above.
$\overline{z}$ will have constant length and will contain the following instructions:

- Use f to find n and t.

- Use s and t to compute $\mid y_\gamma \mid = s \log 2t + s + c_2$.

- Use $y_{\rho'}$ to compute ρ' and y_γ to compute γ.

- Express f as $f = \bigwedge_i D_i$ and find $i_1 = min\{i : \rho'(D_i) \neq 1\}$.

- Use D_{i_1} and γ_1 to find

$$dom(\pi_1) = \{\text{variables in } D_{i_1}, \text{ corresponding to non-stars in } \gamma_1\}$$

  Recall that γ_1 is just the first t variables in γ so γ_1 is given once γ has been found.
  Note:

$$\pi_1 = \rho'|_{dom(\pi_1)} \text{ as } \rho' = \rho\tilde{\pi}_1\tilde{\pi}_2 \ldots \tilde{\pi}_{k-1}\sigma$$

  .

- Build π_1 as follows:

$$\pi_1(i) = \begin{cases} * & i \notin dom(\pi_1) \\ \gamma_j \oplus \tilde{\pi}_1(i) & x_i = j^{th} \text{ variable in } D_{i_1} \end{cases}$$

- Let

$$i_2 = \min\{i : \rho\pi_1\tilde{\pi}_2\tilde{\pi}_3 \ldots \tilde{\pi}_{k-1}\sigma(D_i) \neq 1\}.$$

  As above, find $dom(\pi_2)$ and build π_2. Continuing in this manner, build $\pi_3, \ldots, \pi_{k-1}, \sigma$. (Recall that s is given so we know when σ has been found.)

- Finally,

$$\rho = \rho'|_{\{1 \ldots n\} \setminus dom(\pi_1 \ldots \pi_{k-1}\sigma)}.$$

Thus, using $\overline{z}y_{\rho'}y_\gamma$ we can find ρ and we have shown that

$$\begin{aligned} K(\rho|f, l, s) &\leq \log \binom{n}{l-s} + n - l + s + c_1 + s \log 2t + s + c_2 \\ &= \log \binom{n}{l-s} + n - l + s \log 8t + c \end{aligned}$$

which completes the proof.