Theorem \[ \text{For } k = n^{\frac{1}{4}}, \text{ } k\text{-clique requires monotone circuits of size } n^{\Omega(\sqrt{l})} \]

Remark \ If this could be done for every monotone problem in NP, this would imply \( P \neq NP \).

A slice function \( f \) is one function such that for all \( n \), there exists \( m \) such that

\[
\begin{align*}
    f(x) &= 0 \text{ if } \sum x_i < m \\
    f(x) &= 1 \text{ if } \sum x_i > m
\end{align*}
\]

Note that all slice functions are monotone. There are slice functions that are NP-complete. One can prove that the monotone circuit complexity of a slice function \( f \) is not much greater than the (non-monotone) circuit complexity of \( f \). Thus we may expect that it is not always easy to prove lower bounds on monotone circuit complexity.

Basic idea: Every monotone circuit for \( k \)-clique \( C \) is "approximated" by a monotone depth 2 circuit \( C' \). The circuit has \( m \) "\&" gates with fan in \( \leq \left( \frac{l}{2} \right) \) on the first level, each detects a clique of size no more than \( l(= \sqrt{k}) \). All the outputs of the \( m \) "\&" gates connected to an "\lor" gate. (The circuit just check the presence, but not absence).

What we will show is
1. If $C$ is a small monotone circuit, then for most $k$-cliques $G$, $\tilde{C}(G) \leq C(G)$ and for most complete $(k-1)$-partite graphs $G$, $\tilde{C}(G) \geq C(G)$

2. Every approximator either outputs 0 on most $k$-cliques or outputs 1 on most $(k-1)$-partite graphs

First we are going to prove the following lemma:

**Sunflower Lemma** Let $F$ be a collection of sets, each of size $\leq l$. If $|F| > (p - 1)^l l!$, then there is a sunflower with $p$ petals. (A sunflower with $p$ petals is a collection of sets $S_1, \ldots, S_p$ such that for all $i < j$ and $k < l$, $S_i \cap S_j = S_k \cap S_l$.)

**Proof** By induction on $l$:

**Basis:** $l = 1$, choose $p$ disjoint singletons.

**Induction:** Let $M$ be a maximal collection of disjoint sets in $F$, $M = \{A_1, A_2, \ldots, A_n\}$. Let $S = \bigcup_{i=1}^n A_i$. If $r \geq p$ we have done.

Otherwise $|S| \leq l(p-1)$. Since $M$ is maximal, $S$ intersects every element of $F$. There is some $i \in S$, such that $i$ is in $\geq 1/(p-1)l$ of the elements of $F$.

Let $F' = \{B - \{i\} : i \in B \in F\}$. $|F'| \geq |F|/(p-1)l > \frac{(p-1)l}{(p-1)(l-1)!}$

Let $B_1 - \{i\}, B_2 - \{i\}, \ldots, B_p - \{i\}$ be a sunflower in $F'$. Then $B_1, B_2, \ldots, B_p$ is our sunflower.

We will build $\tilde{C}$ from $C$, Starting at the leaves, that is, for each gate $g$ of $C$, consider the function computed at $g$, we will build an approximator for that function.

- If $g$ is a leaf, we are done.

- If $g$ is an $\lor$ gate, $g = h_1 \lor h_2$, define $g = h_1 \cup h_2$ as follows:
1. If \( h_1 = \{S_1, S_2, \ldots, S_m\} \), \( h_2 = \{T_1, T_2, \ldots, T_m\} \) (\( S_i, T_j, 1 \leq i \leq m_1, 1 \leq j \leq m_2 \) are sets of vertices (each set of size at most \( l \)) and \( |h_1 \cup h_2| \leq m \)), then set \( g = h_1 \cup h_2 \). Otherwise \( h_1 \cup h_2 \) contains a sunflower of \( p \) petals. Replace those \( p \) sets with their center.

2. If there are still more than \( m \) sets, repeat it until there are no more sunflowers.

- If \( g = h_1 \land h_2 \), define \( h_1 \land h_2 \) as follows:

  Let \( h_1 = \{S_1, S_2, \ldots, S_m\} \), \( h_2 = \{T_1, T_2, \ldots, T_m\} \), \( h_1 \land h_2 = \{S_i \land T_j : 1 \leq i \leq m_1, 1 \leq j \leq m_2\} \)

1. Consider \( \{S_i \cup T_j : 1 \leq i \leq m_1, 1 \leq j \leq m_2\} \)
2. remove any such \( S_i \cup T_j \), such that \( |S_i \cup T_j| > l \),
3. replace sunflowers with their centers (plucking), until there are no more sunflowers (so there are \( \leq m \) sets remaining).

In what follows, we will use the following values for \( l, p, \) and \( m \): \( l = \sqrt{k}, p = 10\sqrt{k}\log n, m = (p - 1)^{l!!} \)

\hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm}

| \textbf{Lemma 1} | Every \( \tilde{C} \) is either identically 1, or it outputs 1 on at least half of the \((k-1)\)-coloured graphs |

\hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm}

| \textbf{Lemma 2} | For all but at most \( \text{Size}(C) \cdot m^2 \left( \frac{n - l - 1}{k - l - 1} \right) \leq \text{Size}(C) \cdot m^2 \left( \frac{k}{n} \right)^{l+1} \left( \frac{n}{k} \right) \) |

\hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm}

\( k \)-cliques \( G, \text{C}(G) \leq \tilde{C}(G) \)
Lemma 3  For all but at most

\[ \text{Size}(C) \cdot \frac{m^2}{2p} (k - 1)^n \]

(k - 1)-partite graphs \( G \), \( C(G) \geq \tilde{C}(G) \)

Theorem  For \( k \leq n^{1/4} \), monotone circuits for \( k \)-clique need size

\[ n^{\Omega(\sqrt{k})} \]

Before proving these lemmas, we will first see how they lead us to our theorem.

Proof  By Lemma 1, we have 2 cases, if \( C \) is identically 1, then Lemma 2 says that

\[ \text{Size}(C) m^2 \left( \frac{k}{n} \right)^{l+1} \left( \frac{n}{k} \right) \geq \left( \frac{n}{k} \right). \]

(That is, the set of cliques such that \( C(g) > \tilde{C} = 0 \) is the set of all cliques.)

So

\[ \text{Size}(C) \geq \left( \frac{n}{k} \right)^{l+1} \geq \frac{(n^{3/4})^{\sqrt{k}}}{\left( 10\sqrt{k} \log n - 1 \right)^{\sqrt{k}} \cdot \sqrt{k}} \geq \frac{n^{\frac{3}{4}\sqrt{k} - \frac{1}{2}\sqrt{k}}}{100} = n^{\Omega(\sqrt{k})} \]
If case 2 holds, then by lemma 3,

\[ \text{Size}(C) \cdot \left( \frac{m^2}{2^p} \right) (k - 1)^n \geq \frac{1}{2} (k - 1)^n \]

\[ \text{Size}(C) \geq \frac{1}{2} \cdot \frac{2^p}{m^2} \geq \frac{1}{2} \cdot \frac{2^{(10\sqrt{k} \log n)}}{(10\sqrt{k} \log n - 1)\sqrt{k} \cdot \sqrt{k}} \geq \frac{1}{2} \cdot \frac{n^{10\sqrt{k} - \frac{1}{2}\sqrt{k}}}{100} = n^{\Omega(\sqrt{k})} \]

Now we are going to prove the lemmas:

**Proof of Lemma 1**

\( \tilde{C} = \bigvee_{i=1}^{r} \text{(clique on set } A_i), |A_i| \leq l, 1 \leq i \leq r \)

If \( r = 0 \), then claim holds.

If \( r = 1 \), Let \( A_1 \) be given, then \( C_1 \) accepts \( G \) iff there is a clique on \( A_1 \) iff all vertices in \( A_1 \) have different colours.

Consider a randomly chosen \((k - 1)\)-coloured graph, Probability that \( G \) is not accepted

\[ = \text{Prob}[\exists i, j \in A_1, i, j \text{ have the same colour}] \]

\[ \leq \binom{l}{2} \frac{1}{k-1} \]

\[ = \frac{\sqrt{k}(\sqrt{k} - 1)}{2} \cdot \frac{1}{k-1} \]

\[ = \frac{k - \sqrt{k}}{k-1} \cdot \frac{1}{2} < \frac{1}{2} \]

If \( r > 1 \), the result also holds, because \( \tilde{C} \) accepts at least as many \( G \) as in the case above.

**Proof of Lemma 2**

If \( G \) is the \( k \)-clique, such that

\( C(G) = 1 \) but \( \tilde{C}(G) = 0 \)

then it must be the case that there is some gate \( g \) in \( C \) such that \( g = h_1 \lor h_2 \) and in \( \tilde{C} \) the approximator built for \( g \) rejects \( G \) but the approximator for
one of \( h_1, h_2 \) accepts \( G \), or \( g = h \land h_2 \) and both \( h_1 \) and \( h_2 \) accept \( G \) but \( g \) rejects.

We will show that any such \( g \) can affect no more than

\[
m^2 \left( \frac{k}{n} \right)^{l+1} \binom{n}{k}
\]

cliques.

Note that if our approximator on \( h_1 \) accepts \( G \), that means that there is some clique \( A \) included in the clique in \( G \), that is detected by our approximator for \( h_1 \). But our approximator for \( h_1 \lor h_2 \) looks for a clique on some subset of \( A \).

So in case 1, our approximator for \( g \geq h_1 \cup h_2 \). So the first problem gate is not of type 1.

Now consider \( g = h_1 \land h_2 \), Let \( A \) be the set of vertices containing the \( k \)-clique \( G \). We are assuming that there is some \( S_i \subseteq A \) and some \( T_j \subseteq A \), so that \( \tilde{h}_1 \) detects a clique on \( S_i \) and \( \tilde{h}_2 \) detects a clique on \( T_j \). But our approximator \( \tilde{g} \) rejects \( A \). Note that in \( G \), there is a clique on \( S_i \cup T_j \). It must be the case that \( |S_i \cup T_j| > l \), because otherwise \( \tilde{g} \) accepts \( G \). Note that there are at most \( m^2 \) such \( (s_i, t_j) \) pairs associated with gate \( g \).

There are no more than \( \binom{n-l-1}{k-l-1} \) such \( k \)-cliques in \( G \) for fixed \( S_i \) and \( T_j \). Thus the number of such \( k \)-cliques is no more than

\[
m^2 \binom{n-l-1}{k-l-1} \leq m^2 \left( \frac{k}{n} \right)^{l+1} \binom{n}{k}
\]

\[1\]

**Proof of Lemma 3**  If \( G \) is a \((k-1)\)-coloured graph such that

\[ C(G) = 0 \quad \text{but} \quad \bar{C}(G) = 1 \]

then it must be the case as above

1. \( g = h_1 \lor h_2 \) such that \( \bar{g} \) accepts \( G \) but \( \tilde{h}_1 \) and \( \tilde{h}_2 \) reject \( G \).

2. \( g = h_1 \land h_2 \) such that \( \bar{g} \) accepts \( G \) but one of \( \tilde{h}_1 \) or \( \tilde{h}_2 \) reject \( G \).
We will show that any such \( g \) can affect no more than

\[
\frac{m^2}{2^p} (k-1)^n
\]

of the \((k-1)\)-coloured graphs.

First consider \( g = h_1 \lor h_2 \). So \( \tilde{g} \) is built from \( \tilde{h}_1 \) and \( \tilde{h}_2 \) from plucking.

Let

\[
\tilde{h}_1 = \sqrt{\text{clique on } S_1, S_2, \ldots, S_{m_1}}
\]

\[
\tilde{h}_2 = \sqrt{\text{clique on } T_1, T_2, \ldots, T_{m_2}}
\]

\( \tilde{h}_1 \) and \( \tilde{h}_2 \) each reject \( G \) iff each \( S_i \) and \( T_j \) are not “properly coloured”.

This means that there is some sunflower in the list \( S_1, S_2, \ldots, S_{m_1} \) and \( T_1, T_2, \ldots, T_{m_2} \), such that \( G \) properly colours the center but \( G \) does not properly colour any of the petals.

Let \( A_1, A_2, \ldots, A_p \) be the sets in this sunflower.

\[
\text{Prob}[\text{none of } A_1, A_2, \ldots, A_p \text{ is properly coloured but the centre is}]
\]

\[
\leq \text{Prob}[A_1, A_2, \ldots, A_p \text{ is not properly coloured}\mid \text{the centre is properly coloured}]
\]

\[
= \prod_{i=1}^{n} \text{Prob}[A_i \text{ is not properly coloured}\mid \text{the centre is properly coloured}]
\]

\[
\leq \prod_{i=1}^{n} \text{Prob}[A_i \text{ is not properly coloured}] \leq \frac{1}{2^p}
\]

And at last, if we notice that we will do plucking for \( m^2 \) time, the result follows naturally.  

[1]