Lecture No. 15

Date: 03-23-95
Scribe: Ramkrishna Chatterjee

We would like to show that monotone circuits for 3-clique require approximately n^3 gates. Here n is the number of vertices in the given graph. We do this by proving a lower bound on the minimum size of a cover (defined below) required to cover a subset of all the monotone functions defined on the power set of a subset (see below) of zeros of the 3-clique function. Let us make these notions precise using the following definitions.

Let n be the number of vertices in the given graph. The input is a $\binom{n}{2}$ bit vector x where x_i is 1 if and only if edge e_i belongs to the given graph. Let us call such an input vector an edge incidence vector.

Definition 1 *Let $U = \{x | x$ is an edge incidence vector for a complete bipartite graph $\}$. So $\forall x \in U \ 3clique(x) = 0$.*

Definition 2 $\Omega_{FILTERS} = \{\omega | \omega : 2^U \rightarrow \{0,1\}$ *and ω is monotone and $\omega(\phi) = 0$ and ω defines a 1 of 3-clique* $\}$

Definition 3 *Let $C \subseteq 2^U \times 2^U$. C is called a cover for $\Omega_{FILTERS}$ if and only if for each $\omega \in \Omega_{FILTERS}$, $\exists (f, g) \in C$ such that*

$$\omega(f) \wedge \omega(g) \neq \omega(f \cap g)$$

Definition 4 *Let $S \subseteq U$. The measure of S is defined as*

$$\mu(S) = \frac{|S|}{|U|}$$

Definition 5 *Let us define the size of a minimum cover as follows.*

$$\hat{\rho}_{FILTERS}(3clique) = \min\{|C| \,|\, C \text{ is a cover for } \Omega_{FILTERS}\}$$

Claim 1 $\hat{\rho}_{FILTERS}(3clique) \geq \frac{\binom{n}{3}}{18^2 \log n^4}$.

Proof: Let C be a given cover of size t. If t is less than the above bound, we will show that there exists a member ω_1 of $\Omega_{FILTERS}$, which is not covered by C (i.e. $\forall (f,g) \in C\, \omega_1(g) \wedge \omega_1(f) = \omega_1(g \cap f)$).

We construct ω_1 as follows.
First we construct a set Q as follows:
Begin

 $Q = U$
 Initially all $(g,h) \in C$ are "unmarked".
 Repeat
  If there is some "unmarked" $(g,h) \in C$
  such that $\mu(g \cap Q) \geq \mu(Q) - \frac{1}{n^3}$
  (or $\mu(g \cap Q) \geq \mu(Q) - \frac{1}{n^3}$)
  then
   $Q = Q \cap g$ (respectively, $Q = Q \cap h$)
   mark (g,h)
 Until no more pairs can be marked

End.

Remark: $\mu(Q) \geq 1 - \frac{t}{n^3}$. This is because the loop executes at most t times, and in each iteration $\mu(Q)$ reduces by $\frac{1}{n^3}$.

Definition 6 *Let e_i, e_j, e_k be a triangle. Here e_i, e_j and e_k are edges. For each edge e_i $\left(1 \leq i \leq \binom{n}{2}\right)$, let $X_i = \{y \mid y \in U \text{ and } e_i \in y\}$. Let $\omega_\triangle : 2^U \rightarrow \{0,1\}$.*

$$\omega_\triangle(g) = 1,\ if\ X_l \cap Q \subseteq g\ for\ l \in \{i,j,k\}.\ Otherwise\ it\ is\ 0$$

Now we try to find a $\omega_\triangle$ which belongs to $\Omega_{FILTERS}$ and is consistent with C (i.e not covered by C). This will show that C fails to cover $\Omega_{FILTERS}$ and hence give us a lower bound for t.

Claim 2 *Any $\omega_\triangle$ is monotone.*

Proof: Obvious from the definition of $\omega_\triangle$.

Claim 3 *$\omega_\triangle$ is consistent with any "marked" pair (g,h).*

Proof : $\omega_\triangle$ is monotone. This means the only way it can be inconsistent is :

$$\omega_\triangle(h) = \omega_\triangle(g) = 1 \, and \, \omega_\triangle(g \cap h) = 0$$

Since (g,h) is "marked", $Q \subseteq g$ (or $Q \subseteq h$). Now suppose $\omega_\triangle(h) = 1$. This means $\exists l \in \{i,j,k\}$ such that $X_l \cap Q \subseteq h$. But $Q \subseteq g$. So $X_l \cap Q \subseteq g$ and therefore $X_l \cap Q \subseteq g \cap h$. Hence $\omega_\triangle(g \cap h) = 1$.

So we need to find a $\omega_\triangle$ which is consistent with the "unmarked" pairs in C. This is made possible by the following lemma.

Definition 7 *Let* $H_g = \{e_i \, : X_i \cap Q \subseteq g \cap Q\}$.

Lemma 1 *If* $|H_g| > 2r^2$ *then* $\mu(g \cap Q) \geq \mu(Q) - \frac{1}{2^r}$.

Proof: Suppose $|H_g| > 2r^2$. Let us view each edge as a subset consisting of two vertices. The "Sunflower" lemma (proved in the next lecture) implies that one of the following two holds (these two situations are not mutually exclusive):

1. H_g contains at least r edges which are incident on the same vertex.
2. H_g has at least r disjoint edges (i.e no two edges are incident on a common vertex).

First consider the first situation. Let for simplicity $e_1, e_2, \cdots, e_r$ be the edges which have a vertex in common. Let $\bigcup_{i=1}^r X_i = X$. Now,

$$\mu(Q) = \mu(Q \cap X) + \mu\left(Q \cap \bar{X}\right)$$

$$\Rightarrow \mu(Q \cap X) = \mu(Q) - \mu\left(Q \cap \bar{X}\right)$$

$$But, \; \mu\left(Q \cap \bar{X}\right) \leq \mu\left(\bar{X}\right)$$

$$\Rightarrow \mu(Q \cap X) \geq \mu(Q) - \mu\left(\bar{X}\right)$$

$$Also, \; \mu\left(\bar{X}\right) = \frac{2^{n-r-1} - 1}{2^{n-1} - 1} \leq \frac{1}{2^r}$$

$$Further, \; g \cap Q \supseteq Q \cap X$$

$$\Rightarrow \mu(g \cap Q) \geq \mu(Q \cap X) \geq 1 - \frac{1}{2^r}$$

We can argue similarly for the second situation.

Let $r = 3\log n$. Thus for each "unmarked" pair (g,h),

$$\mu(g \cap Q) < \mu(Q) - \frac{1}{n^3} = \mu(Q) - \frac{1}{2^r}$$

So by the above lemma $|H_g| \leq 18(\log n)^2$ $(|H_h| \leq 18(\log n)^2)$.

Consider any $\omega_\triangle$. Suppose for a "unmarked" pair (g,h)

$$\omega_\triangle(g) \wedge \omega_\triangle(h) \neq \omega_\triangle(g \cap h)$$

$$\Rightarrow \omega_\triangle(g) = \omega_\triangle(h) = 1 \; and \; \omega_\triangle(g \cap h) = 0$$

As $\omega_\triangle$ is monotone. This also means $\exists e_i, e_j \in \triangle$ such that $e_i \in H_g$ and $e_j \in H_h$. If $e_i = e_j$, $\omega_\triangle(g \cap h) = 1$. So $e_i \neq e_j$. Hence at most $18^2(\log n)^4$ $\omega_\triangle$'s can be inconsistent with (g,h). So the number of $\omega_\triangle$'s which are inconsistent with at least one of the "unmarked" pairs is $\leq t(18)^2(\log n)^4$. Since $t(18)^2(\log n)^4 < \binom{n}{3}$, there exists a $\triangle$ T such that ω_T is consistent with C. Let $\omega_1 = \omega_T$.

Claim 4 $\omega_1(\phi) = 0$.

Proof: Let $e_i \in T$. $\mu(X_i) \geq \frac{1}{2}$. Also for large enough n, $\mu(Q) \geq 1 - \frac{t}{n^3} > \frac{1}{2}$. $\Rightarrow \mu(Q \cap X_i) \neq 0$. $\Rightarrow Q \cap X_i \neq \phi$. The claim now follows by the definition of $\omega_\triangle = \omega_T$.

Claim 5 ω_1 *defines a 3-clique.*

Proof: Obvious from the construction.

So $\omega_1 \in \Omega_{FILTERS}$ and it is not covered by C. This is a contradiction. So t must be greater than or equal to the required bound. As shown in the previous lecture the minimum monotone circuit size is greater than or equal to $\hat{\rho}_{FILTERS}$. Hence the minimum monotone circuit size for computing 3-clique is $\geq \frac{\binom{n}{3}}{18^2 \log n^4}$.