We would like to show that monotone circuits for 3-clique require approximately $n^3$ gates. Here $n$ is the number of vertices in the given graph. We do this by proving a lower bound on the minimum size of a cover (defined below) required to cover a subset of all the monotone functions defined on the power set of a subset (see below) of zeros of the 3-clique function. Let us make these notions precise using the following definitions.

Let $n$ be the number of vertices in the given graph. The input is a $\binom{n}{2}$ bit vector $x$ where $x_i$ is 1 if and only if edge $e_i$ belongs to the given graph. Let us call such an input vector an edge incidence vector.

**Definition 1** Let $U = \{x|x$ is an edge incidence vector for a complete bipartite graph $\}$. So $\forall x \in U \text{ } 3\text{clique}(x) = 0$.

**Definition 2** $\Omega_{\text{FILTERS}} = \{\omega|\omega : 2^U \rightarrow \{0,1\}$ and $\omega$ is monotone and $\omega(\phi) = 0$ and $\omega$ defines a 1 of 3-clique $\}$

**Definition 3** Let $C \subseteq 2^U \times 2^U$. $C$ is called a cover for $\Omega_{\text{FILTERS}}$ if and only if for each $\omega \in \Omega_{\text{FILTERS}}$, $\exists (f, g) \in C$ such that $\omega(f) \land \omega(g) \neq \omega(f \cap g)$

**Definition 4** Let $S \subseteq U$. The measure of $S$ is defined as

$$\mu(S) = \frac{|S|}{|U|}$$

**Definition 5** Let us define the size of a minimum cover as follows.

$$\hat{\rho}_{\text{FILTERS}}(\text{3clique}) = \min\{|C||C$ is a cover for $\Omega_{\text{FILTERS}}\}$$

**Claim 1** $\hat{\rho}_{\text{FILTERS}}(\text{3clique}) \geq \frac{\binom{3}{2}}{18^4 \log n^\tau}$.  

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Proof: Let \( C \) be a given cover of size \( t \). If \( t \) is less than the above bound, we will show that there exists a member \( \omega_1 \) of \( \Omega_{\text{FILTERS}} \), which is not covered by \( C \) (i.e. \( \forall (f, g) \in C \) \( \omega_1 (g) \land \omega_1 (f) = \omega_1 (g \cap f) \)).

We construct \( \omega_1 \) as follows.

First we construct a set \( Q \) as follows:

Begin

\[
Q = U
\]

Initially all \((g, h) \in C\) are “unmarked”.

Repeat

If there is some “unmarked” \((g, h) \in C\)

such that \( \mu (g \cap Q) \geq \mu (Q) - \frac{1}{n^3} \)

(or \( \mu (g \cap Q) \geq \mu (Q) - \frac{1}{n^3} \))

then

\[
Q = Q \cap g \text{ (respectively, } Q = Q \cap h) \]

mark \((g, h)\)

End.

Remark: \( \mu (Q) \geq 1 - \frac{4}{n^3} \). This is because the loop executes at most \( t \) times, and in each iteration \( \mu (Q) \) reduces by \( \frac{1}{n^3} \).

Definition 6 Let \( e_i, e_j, e_k \) be a triangle. Here \( e_i, e_j \) and \( e_k \) are edges. For each edge \( e_i \) \((1 \leq i \leq \left(\begin{array}{c} n \\ 2 \end{array}\right))\), let \( X_i = \{ y \mid y \in U \text{ and } e_i \in y \} \). Let \( \omega_\Delta : 2^U \rightarrow \{0, 1\} \).

\[
\omega_\Delta (g) = 1, \text{ if } X_i \cap Q \subseteq g \text{ for } l \in \{i, j, k\}. \text{ Otherwise it is 0}
\]

Now we try to find a \( \omega_\Delta \) which belongs to \( \Omega_{\text{FILTERS}} \) and is consistent with \( C \) (i.e not covered by \( C \)). This will show that \( C \) fails to cover \( \Omega_{\text{FILTERS}} \) and hence gives us a lower bound for \( t \).

Claim 2 Any \( \omega_\Delta \) is monotone.

Proof: Obvious from the definition of \( \omega_\Delta \).

Claim 3 \( \omega_\Delta \) is consistent with any “marked” pair \((g, h)\).
Proof: \( \omega_\triangle \) is monotone. This means the only way it can be inconsistent is:

\[
\omega_\triangle (h) = \omega_\triangle (g) = 1 \text{ and } \omega_\triangle (g \cap h) = 0
\]

Since \((g, h)\) is “marked”, \(Q \subseteq g \) (or \(Q \subseteq h\)). Now suppose \(\omega_\triangle (h) = 1\). This means \(\exists l \in \{i, j, k\}\) such that \(X_l \cap Q \subseteq h\). But \(Q \subseteq g\). So \(X_l \cap Q \subseteq g\) and therefore \(X_l \cap Q \subseteq g \cap h\). Hence \(\omega_\triangle (g \cap h) = 1\).

So we need to find a \(\omega_\triangle \) which is consistent with the “unmarked” pairs in \(C\). This is made possible by the following lemma.

**Definition 7** Let \(H_g = \{e_i : X_i \cap Q \subseteq g \cap Q\}\).

**Lemma 1** If \(|H_g| > 2r^2\) then \(\mu (g \cap Q) \geq \mu (Q) - \frac{1}{2^r}\).

**Proof:** Suppose \(|H_g| > 2r^2\). Let us view each edge as a subset consisting of two vertices. The “Sunflower” lemma (proved in the next lecture) implies that one of the following two holds (these two situations are not mutually exclusive):

1. \(H_g\) contains at least \(r\) edges which are incident on the same vertex.
2. \(H_g\) has at least \(r\) disjoint edges (i.e., no two edges are incident on a common vertex).

First consider the first situation. Let for simplicity \(e_1, e_2, \ldots, e_r\) be the edges which have a vertex in common. Let \(\bigcup_{i=1}^r X_i = X\). Now,

\[
\mu (Q) = \mu (Q \cap X) + \mu (Q \cap \bar{X})
\]

\[
\Rightarrow \mu (Q \cap X) = \mu (Q) - \mu (Q \cap \bar{X})
\]

But, \(\mu (Q \cap \bar{X}) \leq \mu (\bar{X})\)

\[
\Rightarrow \mu (Q \cap X) \geq \mu (Q) - \mu (\bar{X})
\]

Also, \(\mu (\bar{X}) = \frac{2^{n-r}-1}{2^{n-1}-1} \leq \frac{1}{2^r}\)

Further, \(g \cap Q \supseteq Q \cap X\)

\[
\Rightarrow \mu (g \cap Q) \geq \mu (Q \cap X) \geq 1 - \frac{1}{2^r}
\]

We can argue similarly for the second situation.
Let \( r = 3\log n \). Thus for each “unmarked” pair \((g, h)\),

\[
mu(g \cap Q) < \mu(Q) - \frac{1}{n^3} = \mu(Q) - \frac{1}{2^r}
\]

So by the above lemma \(|H_g| \leq 18(\log n)^2\) \(|H_h| \leq 18(\log n)^2\).

Consider any \( \omega_\Delta \). Suppose for a “unmarked” pair \((g, h)\)

\[
\omega_\Delta (g) \land \omega_\Delta (h) \neq \omega_\Delta (g \cap h)
\]

\[
\Rightarrow \omega_\Delta (g) = \omega_\Delta (h) = 1 \text{ and } \omega_\Delta (g \cap h) = 0
\]

As \( \omega_\Delta \) is monotone. This also means \( \exists e_i, e_j \in \triangle \) such that \( e_i \in H_g \) and \( e_j \in H_h \). If \( e_i = e_j, \omega_\Delta (g \cap h) = 1 \). So \( e_i \neq e_j \). Hence at most \( 18^2(\log n)^4 \omega_\Delta ’s \) can be inconsistent with \((g, h)\). So the number of \( \omega_\Delta ’s \) which are inconsistent with at least one of the “unmarked” pairs is \( \leq t(18)^2(\log n)^4 \). Since \( t(18)^2(\log n)^4 < \left(\frac{3}{2}\right) \), there exists a \( \triangle T \) such that \( \omega_T \) is consistent with \( C \).

Let \( \omega_1 = \omega_T \).

**Claim 4** \( \omega_1 (\phi) = 0 \).

**Proof:** Let \( e_i \in T. \mu(X_i) \geq \frac{1}{2^r} \). Also for large enough \( n, \mu(Q) \geq 1 - \frac{1}{n^3} > \frac{1}{2^r} \Rightarrow \mu(Q \cap X_i) \neq 0 \Rightarrow Q \cap X_i \neq \phi \). The claim now follows by the definition of \( \omega_\Delta = \omega_T \).

**Claim 5** \( \omega_1 \) defines a 3-clique.

**Proof:** Obvious from the construction.

So \( \omega_1 \in \Omega_{FILTERS} \) and it is not covered by \( C \). This is a contradiction. So \( t \) must be greater than or equal to the required bound. As shown in the previous lecture the minimum monotone circuit size is greater than or equal to \( \hat{\rho}_{FILTERS} \). Hence the minimum monotone circuit size for computing 3-clique is \( \geq \frac{\left(\frac{3}{2}\right)}{18^2 \log n} \).