CS538 Spring 98
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Mar. 13

Classes of Languages, Classes of Functions

Let C be any of our complexity classes, ie, AC^0, TC^0, NC^k, AC^k, L, NL, P, NP, etc.
Let f be a function, s.t., $|f(x)| \leq |x|^{O(1)}$
Define: $\{(x, i, b)$: the ith symbol of $f(x)$ is $b\}$
Note the function f is "computable in C" in the usual, intuitive sense if and only if the set $\{(x, i, b)$: the ith symbol of $f(x)$ is $b\} \in C$

Actually, when C = NP, it is not so clear that this is the "usual, intuitive sense". The class of functions for which $\{(x, i, b)$: the ith symbol of $f(x)$ is b $\}$ is in NP is sometimes called SVNP. Alternatively, we define $\{f : \exists NP$ machine M, s.t., on input x, there is some accepting path of M on x, and all accepting paths output $f(x)\}$.

There are some other important classes of functions related to NP in some natural "intuitive" senses:

Classes of functions related to NP:

- FP^{NP}: the class of functions computable in polynomial time with an "oracle" for NP
- $\#P$: $\{f : \Sigma^* \longrightarrow N$, where $\exists$ an NP machine M, s.t., $f(x)$=$\#acc_M(x)$, where $\#acc_M(x)$ is defined to be the number of accepting computations of M on input x.$\}$
- $\{f : \exists$ a predicate R in P, s.t., $f(x) = Max_{y \in \{0,1\}^{|x|^k}}(R(x, y))\}$
  This class of functions is called OptP.

Last time we introduced $\leq_m^C$ reducibility.
This is the most useful notion of reducibility, but it isn't the most general notion. Sometimes, the more general notions are important.

- $\leq_T^C$, (Turing reducibility) captures the notion of having a free "subroutine", which is also called an "Oracle".

- Oracle Turing Machines have an additional "oracle tape", which alternates between write-only and read-only states. Associated with any oracle machine is a particular problem Y which the oracle can solve for free. While the oracle tape is in write-only state, the TM can at any time enter a query y, ie, write the query string on the oracle tape. At the next step the contents y of the oracle tape will be replaced by a string b such that $(y, b) \in Y$, and the tape will become read-only.

Claim: Multiplication is "not harder than" MAJORITY. See Figure 1

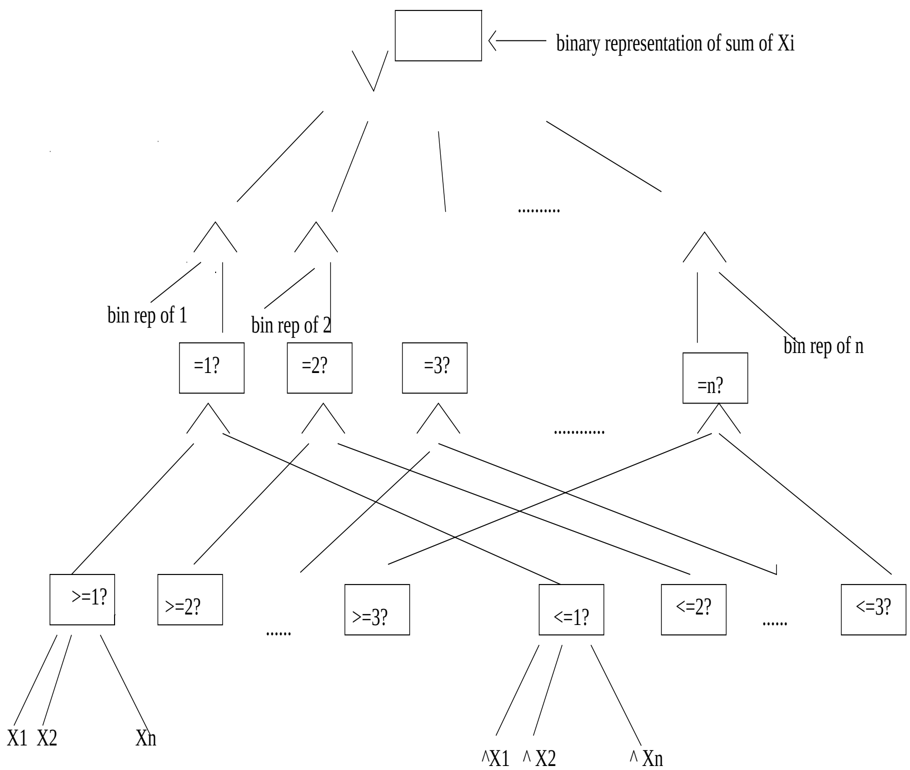

Figure 1: Adding n bits can be done in TC^0

Therefore we get:

MULTIPLICATION$\in TC^0$

MULTIPLICATION$\leq_T^{AC^0}$ MAJORITY

More precisely, in a circuit-based model, we say $A \leq_T^C B$ if we can build "C-circuit" for A using B-gates.

Conversely, MAJORITY$\leq_T^{AC^0}$ MULTIPLICATION. See Figure 2

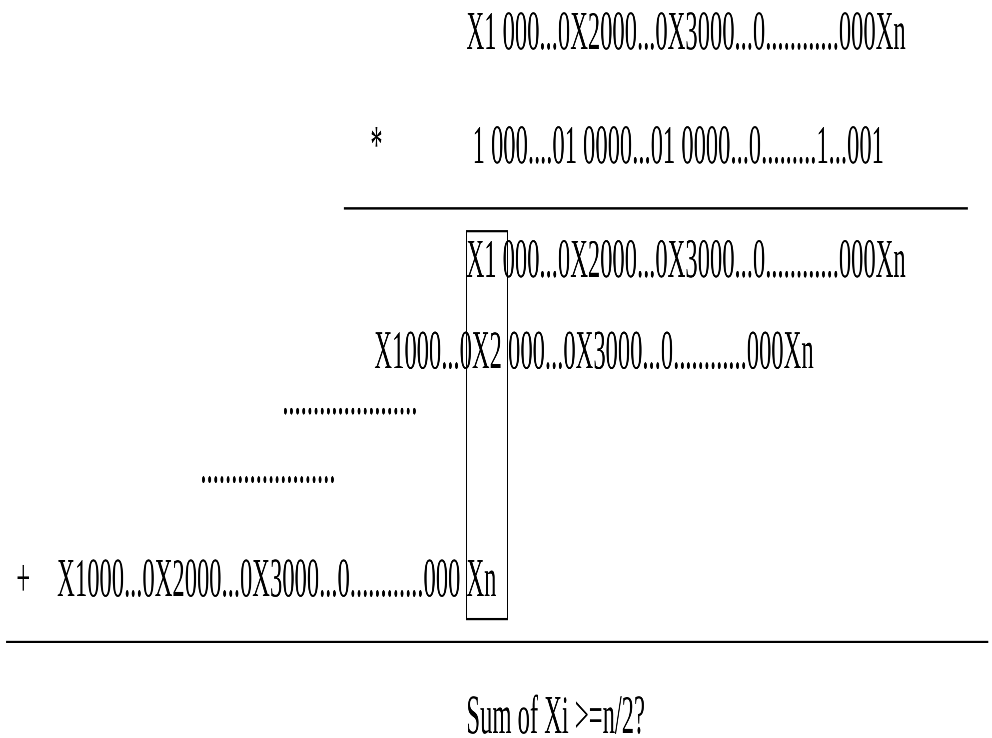

Figure 2: MAJORITY$\leq_T^{AC^0}$ MULTIPLICATION

MULTIPLICATION$\equiv_T^{AC^0}$MAJORITY$\equiv_T^{AC^0}$SORTING$\equiv_T^{AC^0}$
All of these problems are complete for TC^0 under $\leq_T^{AC^0}$ reducibility.
Fact: Most experts believe TC^0 has no complete set under $\equiv_{Many-one}^{AC^0}$ reducibility.