CS 538
Aynur Dayanik
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In this lecture, we will discuss Nondeterministic Time Hierarchy Theorem. Recall the Deterministic Time Hierarchy Theorem.

Theorem: (Deterministic Time Hierarchy Theorem) If t is a time-constructible function, then $DTIME(t(n)) \subset DTIME(t^2(n))$ (also $DTIME(t(n)) \subset DTIME(t(n)\log^2 t(n))$).

Note that $\subset$ is used for proper inclusion.

For the proof of this theorem, we used the diagonalization argument. The basic idea is to reject when the deterministic TM accepts, and to accept otherwise. However, we cannot apply the diagonalization argument directly in the case of nondeterministic machines. There are some accepting paths and some rejecting paths in a computation with nondeterministic Turing machines.

Theorem: (Nondeterministic Time Hierarchy Theorem) Let T be a time constructible function.

If $t(n+1) = o(T(n))$, then $NTIME(t(n)) \subset NTIME(T(n))$.

Corollary: $NTIME(n) \subset NTIME(n \log^* n)$.

Open Question: $DTIME(n) \ =?\ DTIME(nlg^* n)$.

Note that $DTIME(2^{2^{2^n}}) \subset DTIME((2^{2^{2^n}})^2)$.

On the other hand, the following equality is still an open question:

$$NTIME(2^{2^{2^n}}) \ =?\ NTIME((2^{2^{2^n}})^2).$$

Thus, for "small" time bounds the nondeterministic time hierarchy is tighter, and for "large" time bounds the deterministic time hierarchy is tighter.

FACT: If $A \in NTIME(t(n))$, then A is accepted by a Nondeterministic Turing Machine with 2 tapes in time $t(n)$.

Proof of the Nondeterministic Hierarchy Theorem:

Let $M_1,\ M_2,\ \cdots$ be an enumeration of two-tape Turing machines of time complexity $t(n)$.

Let f be a rapidly growing function (such as $f(i,n,s) = 2^{2^{i+n+s}}$) such that the predicate

$$(i,n,s) \longmapsto \begin{cases} 1 & \text{if } M_i \text{ accepts } 1^n \text{ in} \leq \text{s steps} \\ 0 & \text{otherwise} \end{cases}$$

is computable deterministically in time $f(i,n,s)$.

Now let us divide the natural numbers into regions by defining *start* and *end* of each region as follows:

start$(1) = 1$
start$(i+1) =$ end $(i) + 1$
end$(j) = f(i,$ start$(j),\ T($start$(j)))$ where $j = (i, y)$

Note that in region $j = (i, y)$, we try to fool machine M_i, and on input $1^{end(j)}$, a deterministic machine can, in time $T(end(j))$, determine whether M_i accepts $1^{start(j)}$ in at most $T(start(j) - 1)$ steps.

Let us look at the code.

on input 1^n
Find $j = (i, y)$ such that $start(j) \le n \le end(j)$ **(I)**
If $n = end(j)$
then
accept iff **(II)** M_i does not accept $1^{start(j)}$ in $\le T(start(j))$ steps.
else
accept iff **(III)** our universal NTM accepts $(i, 1^{n+1})$ in $T(n)$ steps.

Step (I) takes $O(n)$ time, step (II) takes deterministically $T(n)$ steps, step (III) takes $T(n)$ steps.

Let A be the language accepted by the above algorithm. Clearly, $A \in NTIME(T(n))$. We claim that $A \notin NTIME(t(n))$.

Assume that $A \in NTIME(t(n))$. Let M_i be the nondeterministic machine accepting A in time $t(n)$, namely, $A = L(M_i)$ where $t_i(n) \le t(n)$.

Consider $j = (i, y)$ where y is so big that

$$\forall n \ge j, t(n+1) \le T(n)/i^3.$$

Consider n such that $start(j) \le n \le end(j)$.

$$1^n \in A \quad (1)$$
$$\text{iff } U \text{ accepts } (i,\ 1^{n+1}) \text{ in } \le\ T(n) \text{ steps} \quad (2)$$
$$\text{iff } M_i \text{ accepts } (i, 1^{n+1}) \text{ in } T(n)/i^3 \text{ steps} \quad (3)$$
$$\text{iff } M_i \text{ accepts } 1^{n+1} \text{ in } T(n)/i^3 \text{ steps, } M_i \text{ accepts } 1^{n+1} \text{ in } t(n+1) \text{ steps} \quad (4)$$
$$\text{iff } 1^{n+1} \in A. \quad (5)$$

Therefore, $1^n \in A$ iff $1^{n+1} \in A$. This contradicts the fact that $1^{start(j)} \in A$ if and only if $1^{end(j)} \notin A$. □

Let us check both directions for (2) iff (3) above.

($\Longrightarrow$) If M_i does not accept in $T(n)/i^3$ steps, then M_i does not accept it at all, since $t(n+1) < T(n)/i^3$, so U does not accept $(i, 1^{n+1})$.

($\Longleftarrow$) If M_i does accept in $T(n)/i^3$ steps, then U has time to compute the simulation. □

Next time, we will talk about Immerman-Szelepzeny and Savitch Theorems.