Continuation of NL/poly=UL/poly

These notes continue the ones given on Feb 10, 2000. We will start with the result from the previous day:

$A = \{(G_1, ..., G_m) | \exists i > 0 \quad G_i \text{is min unique} \wedge G_i \in \text{GAP} \wedge \forall j < i \quad G_j \text{ is not min-unique}\} \in UL$

Remember, min-unique means that the shortest path on a graph is unique.

We need to do the following steps to complete the proof of NL/poly=UL/poly:

1. Present a randomized procedure for GAP.
2. Derandomize it.
3. Demonstrate that this completes the proof.

Creating min-unique graph

1. Given a graph G with n vertices, consider an operation of picking a random weight $W(e) \in \{1, 2, ..., 4n^4\}$ for each edge e.

   In this case, weight can be thought of as either a cost associated with traversing that edge, or you could add $W(e) - 1$ additional vertices of degree 2 between the two vertices spanned by edge e.

   Theorem. *With probability $\geq \frac{3}{4}$, the resulting graph is min-unique.*

   Proof. We'll determine the chance that the resulting graph is not min-unique.

   Define the property Bad as follows: an edge e and a vertex x have the property $\text{Bad}(x, e)$ if e is on one min-weight path ¿from $s \to x$ but not another.

   Each edge can contribute to the graph being bad, so the total chance that the graph is not min-unique can be expressed as

$$\begin{aligned} Pr_{\text{Graph is not Min-Unique}} &= \\ Pr(\exists e, x\text{Bad}(x, e)) &\leq \sum_{e,x} Pr(\text{Bad}(x, e)) \\ &= \sum_{e,x} \sum_{w'} Pr(\text{Bad}(x, e)|w')Pr(w') \end{aligned}$$

   In this case, w' is looking over all possible weights except for edge e. This means that $Pr\text{Bad}(x, e) = \sum_{\text{all possible } w'} Pr(\text{Bad}(x, e)|w')Pr(w')$. This is the probability that an edge and vertex are Badgiven a weight w' for that edge, with the probability that we picked w' for that edge.

   □

Claim. $Pr(\textsc{Bad}(x,e)|w') \leq \frac{1}{4n^4}$

Proof. If a (vertex,edge) combination have the property of $\textsc{Bad}(x,e)$, it occurs when $w(e)$ is chosen to be some value r. Any other pick would make that edge worth a different amount, and would either make the path worth less, in which case it could be a unique minimum path (as long as that egde e was not on every min-weight path, but in that case, the event $\textsc{Bad}$would not occur), or the other choice of the weight would cause the path to be more, and it is no longer a minimum path in the graph.

So at most one pick can make $\textsc{Bad}(x,e)$ happen, and therefore $Pr(\textsc{Bad}(x,e)|w') \leq \frac{1}{4n^4}$.

This means that the total probability is

$$\leq \sum_{e,x} \sum_{w'} \frac{1}{4n^4} = \sum_{e,x} \frac{1}{4n^4} \leq \frac{1}{4n} \leq \frac{1}{4} \tag{1}$$

□

2. Derandomize

Claim There is a sequence of n^2 weight functions $< w_1, w_2, ..., w_{n^2} >$, such that $\forall G \quad \exists i \quad s.t. \, w_i(G)$ is min-unique. In this case a weight function is a mapping of weights to edges for a graph.

	G_1	G_2	...	...	$G_{2^{n^2}}$
list of $2^{polynomial}$ weight fns	1	0	0	$\cdots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	...	$\vdots$

Proof. The above table is created by listing all combinations of groups of n^2 weight functions, where a weight function is an assignment of weights to edges. Each cell in the matrix represents the answer to the question, "Does one of the n^2 weight functions create a min-unique graph from G_i?". The answer is "Yes" (1), or "No"(0).

In any location, the probability that none of the w_i are good is $\leq \frac{1}{4^{n^2}}$. So, for every G_i, $\leq \frac{1}{4}^{n^2}$ fraction of the rows in column G_i contain a zero. So the total number of zeros in the matrix is $\leq 2^{n^2} \frac{1}{4}^{n^2}$ number of rows $< \frac{1}{2^{n^2}}$number of rows. This means that some row only has 1's, which means that a sequence of weight functions will work for all graphs. □

3. Now, for the UL/poly algorithm. Given a set of graphs, use the weight function determined from above as the advice sequence, see if

$(w_1(G), w_2(G), ..., w_{n^2}(G)) \in A$.

We have chosen a set of weight functions that for any given graph, one of these will make G min-unique. Therefore there will be an i such that G_i

is min-unique, and all $j < i$ will not be min-unique. Specifically, this will be the first w_i that makes G min-unique.

To see that GAP is in NL/poly, observe that given G and this good set of weight functions, a machine can construct a sequence of graphs (using the weight function to stretch an edge of weight i into a path of length i), and then use the fact that the set A is in UL.

So we have demonstrated that $NL/poly = UL/poly$.