Recitation 7

Internet Technology (Section 01)

Queuing Theory

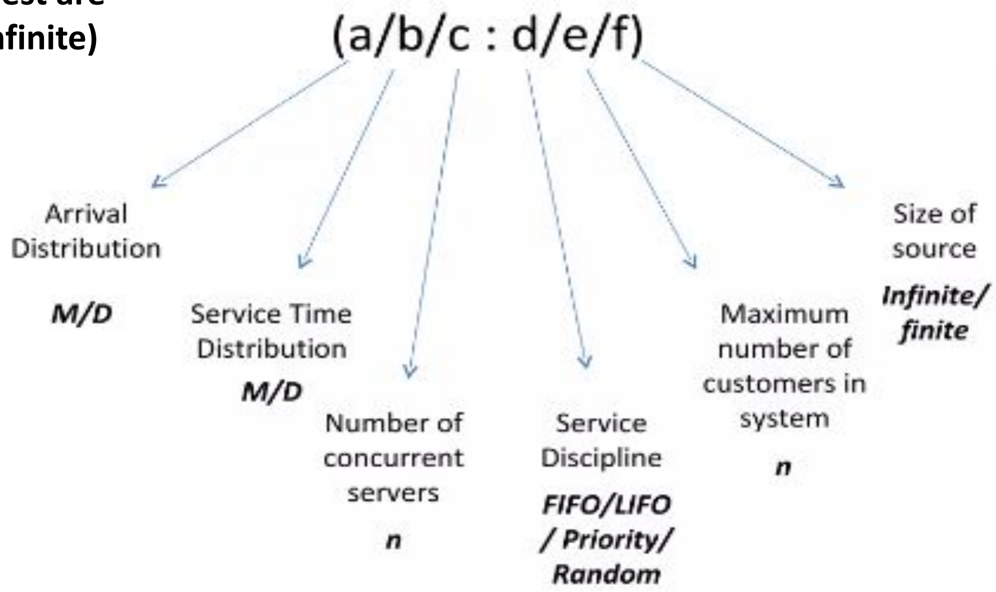
- Study of information, people, or objects through waiting lines (queues)
- Applications
  - Telecommunications
  - Traffic Control
  - Health Services
  - Airport Traffic
  - Predicting computer performance

Queuing Theory for Networks

- View networks as a collection of queues
- Queuing theory allows us to perform probabilistic analysis
  - Average queue length
  - Average queue waiting time
  - Probability the queue is a certain length
  - Probability a packet will be lost

Kendall Notation

Typically first 3 specified, rest are defaulted (FIFO, infinite, infinite)



The M/M/1 Queue Model

- Arrival distribution determined by poisson process
- Exponential service time
- 1 server

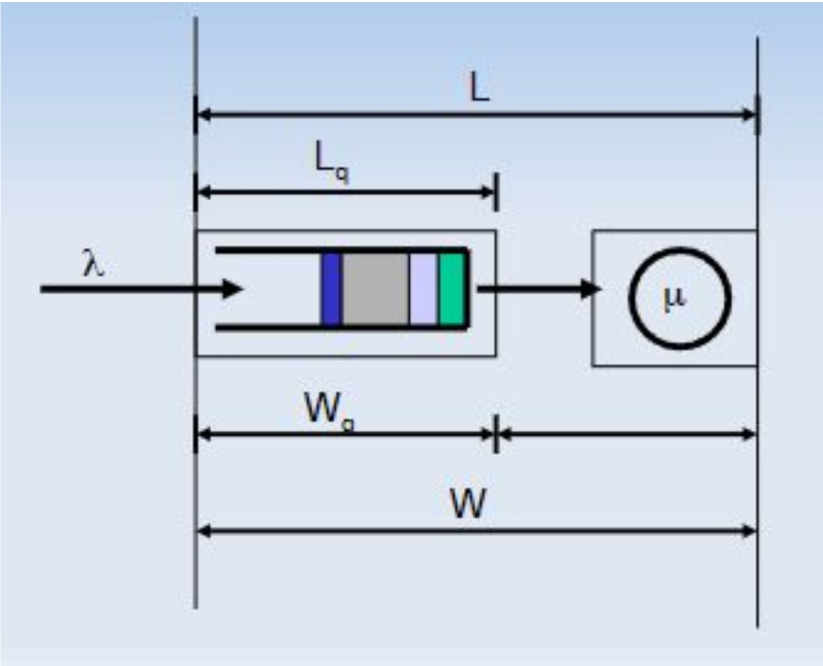
Given:

- **λ**: Arrival rate of jobs
- **μ**: Service rate of the server

Solve:

- **L**: average number of events in Q-ing system
- L_q: average number of events in a queue
- **W**: average waiting time in the whole system
- W_q: average waiting time in a queue

M/M/1 Queue Model Visualized



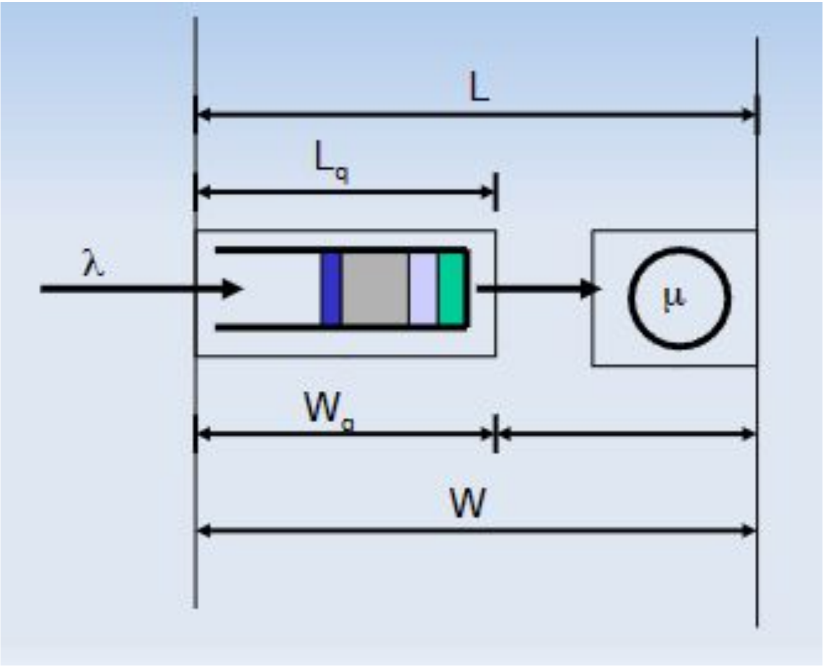
Little’s Law

- Mean # tasks in system = mean arrive rate * mean response time
    - As long as system does not create or destroy tasks

$$\text{Minutes until I get coffee} = \frac{\text{Number of people in line}}{\text{people served per minute}}$$

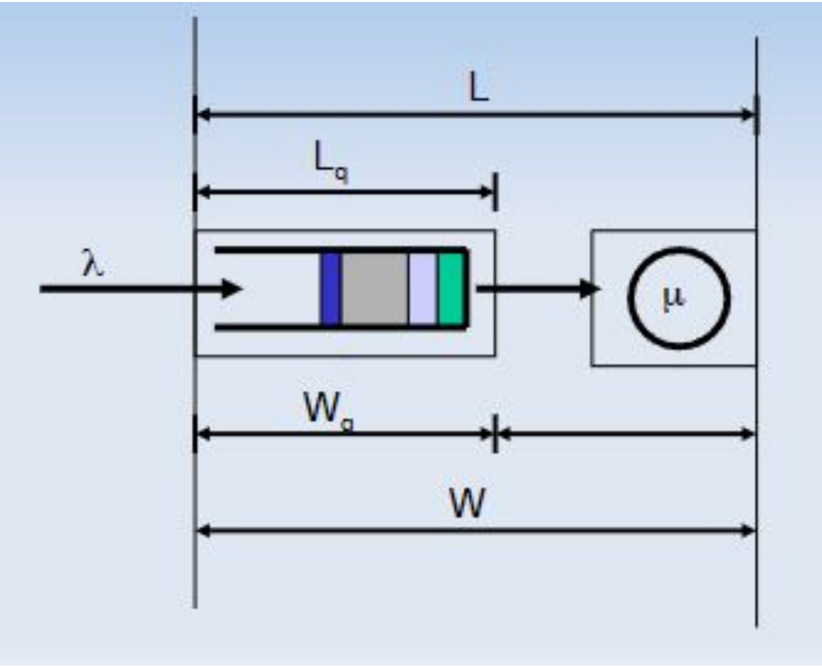
Little’s Law in M/M/1 Queue Model

- $L = \lambda W$
- $L_q = \lambda W_q$
- $W = W_q + 1/\mu$



Analysis of M/M/1 Queue Model

- Goal
  - A closed form expression of the probability of the number of jobs in the queue (Pi) given only λ and μ.
  - If we know any of the 4 unknowns we can find others.

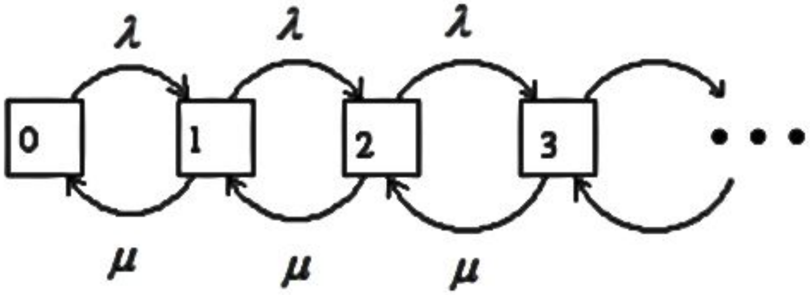


Analysis of M/M/1 Queue Model

Probability of **i** jobs in system: $\mathbf{P_i}$

$$\lambda P_0 = \mu P_1$$

$$(\lambda + \mu) P_n = \lambda P_{n-1} + \mu P_{n+1}$$

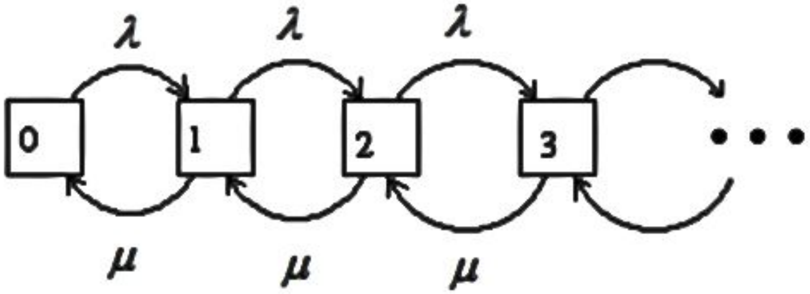


Analysis of M/M/1 Queue Model

$$P_1 = \frac{\lambda}{\mu} P_0,$$

$$P_2 = \left(\frac{\lambda}{\mu}\right)^2 P_0,$$

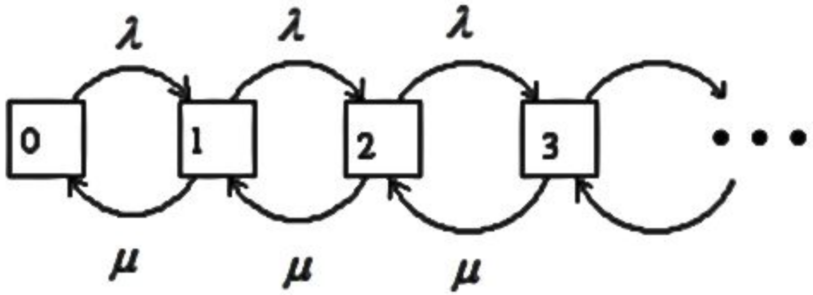
$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$$



Analysis of M/M/1 Queue Model

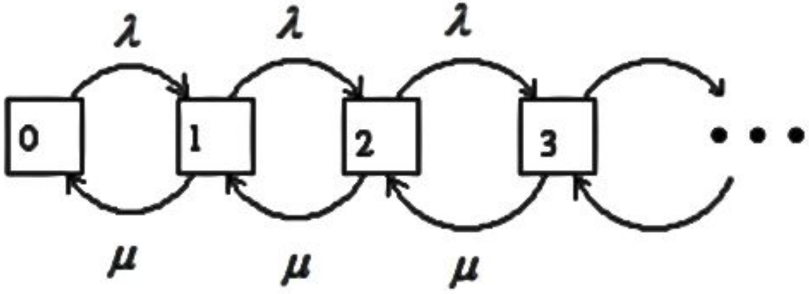
$$\sum_{n=0}^{\infty} P_n = 1,$$

$$P_0 \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n = 1, \quad \Rightarrow \quad P_0 = \frac{1}{\sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n}$$



Analysis of M/M/1 Queue Model

$$\rho = \frac{\lambda}{\mu},$$

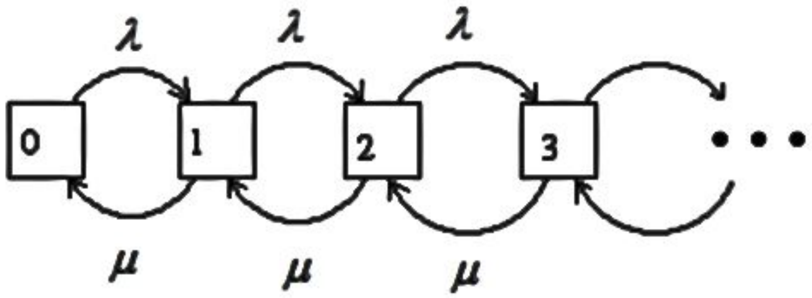


$$\sum_{n=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^{n} = \sum_{n=0}^{\infty}\rho^{n} = \frac{1-\rho^{\infty}}{1-\rho} = \frac{1}{1-\rho}\{\rho < 1\}$$

Analysis of M/M/1 Queue Model

$$P_0 = \frac{1}{\sum_{n=0}^{\infty} \rho^n} = 1 - \rho$$

$$P_n = \rho^n (1 - \rho)$$



Analysis of M/M/1 Queue Model

$$L = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} n\rho^n(1-\rho) = (1-\rho)\rho\sum_{n=1}^{\infty} n\rho^{n-1}$$

$$(1-\rho)\rho\frac{d}{d\rho}\left(\sum_{n=0}^{\infty}\rho^n\right) = (1-\rho)\rho\frac{d}{d\rho}\left(\frac{1}{1-\rho}\right)$$

$$(1-\rho)\rho\left(\frac{1}{(1-\rho)^2}\right) = \frac{\rho}{(1-\rho)} = \frac{\lambda}{\mu-\lambda}$$

Analysis of M/M/1 Queue Model

$$L = \frac{\lambda}{\mu - \lambda}$$

$$W = \frac{L}{\lambda} = \left(\frac{\lambda}{\mu - \lambda}\right)\left(\frac{1}{\lambda}\right) = \frac{1}{\mu - \lambda}$$

$$W_q = W - \frac{1}{\mu} = \left(\frac{\lambda}{\mu - \lambda}\right) - \left(\frac{1}{\mu}\right) = \frac{\lambda}{\mu(\mu - \lambda)}$$

$$L_q = \lambda W_q = \lambda \frac{\lambda}{\mu(\mu - \lambda)} = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

Example 1

On a network gateway, measurements show that the packets arrive at a mean rate of 200 packets per second (pps) and the gateway takes about 2 milliseconds to forward them.

A. (7 points) Assuming an M/M/1 model, what is the probability of buffer overflow if the gateway had only 8 buffers?

$\lambda = 200\text{pps}$

$\mu = 500\text{pps}$

$\rho = 0.4$

$$P(\geq n \text{ jobs in system}) = \sum_{j=n}^{\infty} p_j = \sum_{j=n}^{\infty} (1-\rho)\rho^j = \rho^n$$

P(9+ packets in gateway) $= P_8 + P_9 + P_{10} + P_{11} + \ldots$

$= \rho_8 - \rho_9 + \rho_9 - \rho_{10} + \rho_{10} - \rho_{11} + \rho_{11} - \ldots$

$= \rho_8 = 0.4^8 = 0.00065536 = 6.5\text{x}10^4$

About 6 per 10,000 packets.

Example 2

B. (8 points) How many buffers are needed to keep packet loss below one packet per billion?

$\rho^n \leq 10^{-9}$

or $n > \log(10^{-9}) / \log(0.4)$

$n > -9 / \log(0.4)$

$n > 22.61$, or about 23 buffers